Existence of Extremal Solutions for Functional Difference Equation

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Abstract

Boundary value problem for fourth order difference equation

$$\begin{gathered}\Delta^4 u(t-2) = f(t, u(t), \Delta^2 u(t-1)) \\ u(0) = 0,\ u(N) = 0, \\ u(-1) + u(1) = 0,\ u(N-1) + u(N+1) = 0\end{gathered}$$

is investigated and a theorem for the existence of solutions for the above problem is obtained.

Key words : Difference Equation, Schauder's fixed point theorem, Compact operator.

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1. Introduction

Several authors [1 - 4, 8, 12, 16] have investigated fourth-order boundary value problems of ordinary differential equations. Under some assumptions of the boundedness or growth conditions for *f*, they obtained the existence of solutions. The boundary value problems of functional differential equations have been investigated by [5, 6, 10, 11, 13-15] for its applications in Physics and control theory. The monotone iterative methods have been used a lot for the existence of extremal solutions of BVP of ordinary differential equations. In this article, we shall develop the method to study the fourth order boundary value problems for functional difference equation with the form

$$\left.\begin{gathered}\Delta^4 u(t-2) = f(t, u(t), \Delta^2 u(t-1)), t \in I; \\ u(0) = 0,\ u(N) = 0; \\ u(-1) + u(1) = 0,\ u(N-1) + u(N+1) = 0.\end{gathered}\right\} \tag{1.1}$$

Instcad of growth restriction [4, 16] or strong monotonicity [12] on *f*, we shall show a theorem for existence of solutions between a lower solution and upper solution β under weak monotonicity assumption on *f*.

Assume that X is a Banach space and P is closed cone in X. For $x, y \in X$, the partial order relation $\leq$ with respected to cone P is defined by $x \leq y$ if $y - x \in P$. The topology and ordering

on X are compatible in the sense that $x_n \le y_n$, $x_n \to x^*$, $y_n \to y^*$, $n \to \infty$ then $x^* \le y^*$.
For u_0, $v_0 \in X$, with $u_0 \le v_0$, define order interval $[u_0, v_0] = \{x \in E : u_0 \le x \le v_0\}$.
The following Lemma [7] is used in Theorem 3.1 to show the existence of extremal solutions of BVP (1.1).

Lemma 1.1 [7] : Let *X* be an ordered Banach space induced by *P*. For u_0, $v_0 \in X$, assume $A : [u_0, v_0] \to [u_0, v_0]$ is an operator. Assume that the following hold:

(a) A is nondecreasing operator;

(b) $u_0 \le Au_0$, $Av_0 \le v_0$;

(c) A is continous;

(d) $A([u_0, v_0])$ is pre-compact in *X*.

Then A has maximal fixed point x^* and minimal fixed point x_* in X. Furthermore, the sequence $\{u_n\}$ and $\{v_n\}$ which is defined by

$$u_n = Au_{n-1}, v_n = Av_{n-1}, n = 1, 2, 3, \ldots$$

satisfy,

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots \le v_n \le \ldots \le v_1 \le v_0,$$

$$u_n \to x_* \text{ and } v_n \to x^* \text{ as } n \to \infty.$$

2. Maximum Principles

In this section, two lemmas on maximum principles are presented. Suppose that $I = \{1, 2, 3, \ldots, N-1\}$; $J = \{0, 1, 2, 3, \ldots, N\}$; E is the set of real valued functions defined on J and $E_1 = \{u \in E : u(0) = 0 = u(N)\}$. The space E with the norm $\|x\| = \max_{t \in J} |x(t)|$, $x \in E$, is closed and convex Banach space. Now consider the BVP

$$\left.\begin{array}{r} \Delta^4 u(t-2) - M\Delta^2 u(t-1) = h(t), t \in I; \\ u(0) = 0,\ u(N) = 0; \\ u(-1) + u(1) = 0,\ u(N-1) + u(N+1) = 0. \end{array}\right\} \tag{2.1}$$

Put $-v(t) = \Delta^2 u(t-1) - Mu(t)$, then $v(0) = 0 = v(N)$.

Therefore (2.1) reduces to

$$\left.\begin{array}{r} -\Delta^2 v(t-1) = h(t), t \in I; \\ v(0) = 0,\ v(N) = 0, \end{array}\right\} \tag{2.2}$$

and

$$\left.\begin{array}{r} \Delta^2 u(t-1) - Mu(t) = -v(t), t \in I; \\ u(0) = 0,\ u(N) = 0, \end{array}\right\} \tag{2.3}$$

The solution of (2.2) is given by $v(t) = \sum_{s=1}^{N-1} G(t, s)h(s)$, where $G(t, s)$ is the Green's function for the BVP $-\Delta^2 v(t-1) = 0; v(0) = 0,\ v(N) = 0,$

which is given by

$$G(t,s) = \begin{cases} \frac{t(N-s)}{N}, & t \le s \\ \frac{s(N-t)}{N}, & s \le t. \end{cases}$$

The Green's function G satisfy the property [9] :

$$\sup_{t\in J} \sum_{s=1}^{N-1} |G(t,s)| \le \frac{N^2}{8}. \qquad (2.4)$$

The Green's function $G_1(t, s)$ of BVP

$$\left.\begin{array}{c} \Delta^2 u(t-1) - Mu(t) = 0, t \in I; \\ u(0) = 0, \ u(N) = 0, \end{array}\right\}$$

is given by

$$G(t,s) = \begin{cases} \frac{(\lambda_1^t - \lambda_2^t)(\lambda_1^{s-N} - \lambda_2^{s-N})}{(\lambda_1 - \lambda_2)(\lambda_1^N - \lambda_2^N)}, & t \le s \\ \frac{(\lambda_1^s - \lambda_2^s)(\lambda_1^{t-N} - \lambda_2^{t-N})}{(\lambda_1 - \lambda_2)(\lambda_1^N) - \lambda_2^N)}, & s \le t. \end{cases}$$

where $\lambda_1 = \frac{(M+2)+\sqrt{M(M+4)}}{2}$ and $\lambda_2 = \frac{(M+2)-\sqrt{M(M+4)}}{2}$.

Here λ_1 and λ_2 are the characterstic roots of

$$\Delta^2 u(t-1) - Mu(t) = 0.$$

Therefore the solution of equation (2.3) is

$$u(t) = -\sum_{s=1}^{N-1} G_1(t,s)v(s).$$

For $v \in E$, define an operator $T : E \to E$ as

$$(Tv)(t) = -\sum_{s=1}^{N-1} G_1(t,s)v(s). \qquad (2.5)$$

Now we prove the following lemma.

Lemma 2.1 : A mapping T defined by (2.5) is linear and continuous.

Proof : It is obvious to see that T is linear. For continuity of T, it is sufficient to prove that for $\in > 0$ there is $\delta > 0$ such that $||h|| < \delta$ implies $||Th|| < \in$. We have

$$\begin{aligned} ||Th|| &= \sup_{t\in J} \left| -\sum_{s=1}^{N-1} G_1(t,s) \left(\sum_{v=1}^{N-1} G(s,v)h(s) \right) \right| \\ &\le ||h|| \sup_{t\in J} \sum_{s=1}^{N-1} |G_1(t,s)| \frac{N^2}{8} \\ &\le K||h|| \frac{N^3}{8}, \end{aligned}$$

where K $= \frac{\lambda_1^N}{(\lambda_1-\lambda_2)(\lambda_1^N-\lambda_2^N)}$. Thus for any $\in\, > 0$ choose $\delta < \frac{8\epsilon}{KN^3}$ which serves the purpose.

Now, it is followed by two lemmas for maximum principle.

Lemma 2.2 : Assume that M ≥ 0, $v \in$ E such that

$$-\Delta^2 v(t-1) + Mv(t) \geq 0; \;\; t \in I, \;\; v(0) \geq 0, \;\; v(N) \geq 0, \tag{2.6}$$

then v (t) ≥ 0 for all t $\in$ J.

Proof : Let $m = min_{t\in J} v(t)$. Assume m < 0, then there exists some $t_1 \in I$ such that

$$m = v(t_1) < 0 \text{ and } \Delta v(t_1) = v(t_1+1) - v(t_1) \geq 0. \tag{2.7}$$

From (2.6), $\Delta v(t) - \Delta v(t-1) \leq Mv(t)$. For any integers r_1, r_2 with $r_1 < r_2$, we have

$$\Delta v(r_2) - \Delta v(r_1 - 1) \;\; \leq \;\; \sum_{t=r_1}^{r_2} Mv(t). \tag{2.8}$$

Let $t_0 = inf\{t : 0 < t \leq t_1, v(t) \leq 0\}$. It is clear that $v(t_0) \leq 0$, $v(t) \leq 0$ for $t_0 \leq t \leq t_1$ and $\Delta v(t_0 - 1) = v(t_0) - v(t_0 - 1) < 0$. From (2.8), we have

$\Delta v(t_1) \leq \sum_{t=t_0}^{t_1} Mv(t) + \Delta v(t_0 - 1) < 0,$

which contradicts to (2.7). Hence v (t) ≥ 0 for all t $\in$ J. For M = 0 the following corollary holds.

Corollary 2.1: If $\Delta^2 v(t-1) \leq 0$, $v(0) \geq 0$, $v(N) \geq 0$ for t $\in$ I, then $v(t) \geq 0$ for t $\in$ J.

Lemma 2.3 : Assume that M ≥ 0, $u \in$ E such that

$$\Delta^4 u(t-2) - M\Delta^2 u(t-1) \geq 0, \;\; t \in I;$$
$$u(0) \geq 0, \;\; u(N) \geq 0;$$
$$u(-1) + u(1) \leq 0, \;\; u(N-1) + u(N+1) \leq 0,$$

then $\alpha(t) \leq \beta(t), \; t \in J, \Delta^2\alpha(t-1) \geq \Delta^2\beta(t-1), \; t \in I.$

Proof : Put $v(t) = -\Delta^2 u(t-1)$, then $\Delta^2 v(t+1) = -\Delta^4 u(t-2)$.
Moreover, $v(0) \geq 0$ and $v(N) \geq 0$. By the use of Lemma 2.2, we get $v(t) \geq 0$ for t $\in$ J, so $\Delta^2 u(t-1) \leq 0$, $u(0) \geq 0$, $u(N) \geq 0$, therefore by Corollary 2.1, we have $u(t) \geq 0$ for t $\in$ J. This completes the proof.

3. Existence Theorem

Suppose f : J × R × R → R. First we give definitions of lower and upper solutions for the boundary value problem (1.1).

Definition 3.1: Let α, $\beta \in E$. We say that α is a lower solution of (1.1) if

$$\Delta^4\alpha(t-2) \le f(t,\alpha(t),\Delta^2\alpha(t-1));$$
$$\alpha(0) \le 0,\ \alpha(N) \le 0;$$
$$\alpha(-1)+\alpha(1) \le 0,\ \alpha(N-1)+\alpha(N+1) \le 0.$$

Similarly, β is an upper solution of (1.1) if

$$\Delta^4\beta(t-2) \ge f(t,\beta(t),\Delta^2\beta(t-1));$$
$$\beta(0) \ge 0,\ \beta(N) \ge 0;$$
$$\beta(-1)+\beta(1) \ge 0,\ \beta(N-1)+\beta(N+1) \ge 0.$$

Define $P = \{u \in E_1 : u(t) \ge 0, t \in J \text{ and } \Delta^2 u(t-1) \le 0 \text{ for } t \in I\}$, then P is a cone in E_1. For x, y $\in E_1$, define partial order relation $\le$ with respect to P by $x \le y$ if $y - x \in P$. If x, y $\in E_1$, $x \le y$, then the set $[x, y] = \{u \in E_1 : x \le u \le y\}$ is an order interval in E_1 which may also be written as

$$[x, y] = \{u \in E_1 : x(t) \le u(t) \le y(t), t \in J \text{ and } \Delta^2 x(t-1) \ge \Delta^2 u(t-1) \ge \Delta^2 y(t-1)\}.$$

Theorem 3.1 : Assume that

(H_1) There exists lower solution $\alpha(t)$ and upper solution $\beta(t)$ of (1.1) such that

$$\alpha(t) \le \beta(t),\ t \in J, \Delta^2\alpha(t-1) \ge \Delta^2\beta(t-1),\ t \in I.$$

(H_2) If u_1, $u_2 \in E$, with $\alpha(t) \le u_1(t) \le u_2(t) \le \beta(t)$, then

$$f(t, u_2(t), v(t)) - f(t, u_1(t), v(t)) \ge 0, \text{ for any } v \in E.$$

(H_3) There is a constant $M > 0$ such that

$$f(t, u(t), v_2(t)) - f(t, u(t), v_1(t)) \le M(v_2 - v_1),$$

for any u, v_1, v_2 satisfying $\alpha(t) \le u(t) \le \beta(t)$, $t \in J$ and

$$\Delta^2\beta(t-1) \le v_1 \le v_2 \le \Delta^2\alpha(t-1) \ge, t \in I.$$

(H_4) f is completely continuous mapping.

Then there exists two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ satisfying

$$\alpha_0 \le \alpha_1 \le \dots \le \alpha_n \le \dots \le \beta_n \le \dots \le \beta_1 \le \beta_0,$$

which respectively converges to extremal solutions $\alpha*$ and $\beta*$ of (1.1) in $[\alpha(t), \beta(t)]$ such that

$$\alpha(t) \le \alpha^*(t) \le \beta^*(t) \le \beta(t),\quad t \in J,$$
$$\Delta^2\alpha(t-1) \ge \Delta^2\alpha^*(t-1) \ge \Delta^2\beta^*(t-1) \ge \Delta^2\beta(t-1), t \in I.$$

Proof : For any $u \in E_1$, let

$$Fu(t) = f(t, u(t), \Delta^2 u(t-1)) - M\Delta^2 u(t-1),\ t \in I, \tag{3.1}$$

where $M > 0$ is constant satisfying (H_3). Then $F : E_1 \to E$ is a continuous operator.

The existence problem (1.1) can be written as

$$\left.\begin{array}{c} \Delta^4 u(t-2) - M\Delta^2 u(t-1) = (Fu)(t), t \in I; \\ u(0) = 0,\ u(N) = 0; \\ u(-1) + u(1) = 0,\ u(N-1) + u(N+1) = 0. \end{array}\right\} \tag{3.2}$$

For any $h \in E$, consider the BVP as follows

$$\left.\begin{array}{c} \Delta^4 q(t-2) - M\Delta^2 q(t-1) = h(t), t \in I; \\ q(0) = 0,\ q(N) = 0; \\ q(-1) + q(1) = 0,\ q(N-1) + q(N+1) = 0. \end{array}\right\} \tag{3.3}$$

Then the solution q of (3.3) exists. Define an operator $T : E \to E$ as

$$Th(t) = q(t). \tag{3.4}$$

By Lemma 2.1, T is linear and continuous. Let

$$v(t) = (Au)(t) = [(ToF)u](t),\ t \in J. \tag{3.5}$$

Then $A : E_1 \to E$, and thus from (3.1) to (3.5), the existence problem for (1.1) is changed into the existence for the fixed points of operator A. Now we shall show that A satisfies assumptions in Lemma 1.1. It is obvious from (H_4) and continuity of T that A satisfies (c) and (d) of Lemma 1.1. Now consider the interval

$$[\alpha_0, \beta_0] = \{u \in E : \alpha_0 \le u_0 \le \beta_0\},\ \ \alpha_0 < \beta_0,$$

$\alpha_0(t) = \alpha(t)$ and $\beta_0(t) = \beta(t)$ are lower and upper solutions of (1.1) respectively. Now, for x and y in $[\alpha_0, \beta_0]$ with $x \le y$, define

$$z_x(t) = [(ToF)x](t),\ \ z_y(t) = [(ToF)y](t),\ \ z(t) = z_y(t) - z_x(t).$$

Then we have from (3.1) to (3.3), (H_2) and (H_3),

$$\begin{array}{c} \Delta^4 z(t-2) - M\Delta^2 z(t-1) \ge 0, \\ z(0) = 0 = z(N),\ z(-1) + z(1) = 0 = z(N-1) + z(N+1). \end{array}$$

Lemma 2.3 implies that, $z(t) \ge 0, t \in J$ and $\Delta^2 z(t-1) \le 0, t \in I$. Thus $Ax \le Ay$, so A is nondecreasing operator. Define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ by $\alpha_n = A\alpha_{n-1}$, $\beta_n = A\beta_{n-1}$ $n = 1, 2, 3, \ldots$, $\alpha_0 = \alpha$, $\beta_0 = \beta$. Now $\alpha_1 = A\alpha_0$, therefore α_0, α_1 satisfies

$$\left.\begin{array}{c} \Delta^4 \alpha_1(t-2) - M\Delta^2 \alpha_1(t-1) = f(t, \alpha_0(t), \Delta^2 \alpha_0(t-1)) - M\Delta^2 \alpha_0(t-1); \\ \alpha_1(0) = 0,\ \alpha_1(N) = 0; \\ \alpha_1(-1) + \alpha_1(1) = 0,\ \alpha_1(N-1) + \alpha_1(N+1) = 0. \end{array}\right\} \tag{3.6}$$

Since α_0 is the lower solution of (1.1), we have

$$\left.\begin{array}{c}\Delta^4\alpha_0(t-2) - M\Delta^2\alpha_0(t-1) \le f(t,\alpha_0(t),\Delta^2\alpha_0(t-1)) - M\Delta^2\alpha_0(t-1);\\ \alpha_0(0)\le 0,\ \alpha_0(N)\le 0;\\ \alpha_0(-1)+\alpha_0(1)\le 0,\ \alpha_0(N-1)+\alpha_0(N+1)\le 0.\end{array}\right\} \quad (3.7)$$

Put $z_1(t) = \alpha_1(t) - \alpha_0(t)$, therefore from (3.6) and (3.7), we obtain

$$\begin{array}{c}\Delta^4 z_1(t-2) - M\Delta^2 z_1(t-1) \ge 0;\\ z_1(0)\ge 0,\ z_1(N)\ge 0;\\ z_1(-1)+z_1(1)\le 0,\ z_1(N-1)+z_1(N+1)\le 0.\end{array}$$

Thus Lemma 2.3 implies $z_1(t)\ge 0$ and $\Delta^2 z_1(t-1)\le 0$, i.e. $\alpha_0(t)\le\alpha_1(t)$ and $\Delta^2\alpha_1(t-1)\le\Delta^2\alpha_0(t-1)$. This shows that $\alpha_0 \le A\alpha_0$. Similarly, we can obtain $A\beta_0 \le \beta_0$. Now $\beta_1 = A\beta_0$ implies

$$\left.\begin{array}{c}\Delta^4\beta_1(t-2) - M\Delta^2\beta_1(t-1) = f(t,\beta_0(t),\Delta^2\beta_0(t-1)) - M\Delta^2\beta_0(t-1);\\ \beta_1(0)=0,\ \beta_1(N)=0;\\ \beta_1(-1)+\beta_1(1)=0,\ \beta_1(N-1)+\beta_1(N+1)=0.\end{array}\right\} \quad (3.8)$$

Put $w(t) = \beta_1(t) - \alpha_1(t)$, from (3.6), (3.8), (H_2) and (H_3), we obtain

$$\Delta^4 w(t-2) - M\Delta^2 w(t-1) \ge 0.$$

Moreover, $w(0)\ge 0, w(N)\ge 0$; $w(-1)+w(1)\le 0$, $w(N-1)+w(N+1)\le 0$.

By Lemma 2.3, $w(t)\ge 0$ and $\Delta^2 w(t-1)\le 0$ which implies that $\alpha 1\le\beta 1$. Thus $\alpha_0(t)\le\alpha_1(t) \le \dots \le \alpha_n(t) \le \dots \le \beta_n(t) \le \dots \le \beta_1(t) \le \beta_0(t)$. By Lemma 1.1, A has maximal fixed point $\beta^*(t)$ and minimal fixed point $\alpha^*(t)$. This complete the proof.

4. Example

Consider a BVP

$$\left.\begin{array}{c}\Delta^4 u(t-2) = u^2(t) - \frac{(\Delta^2 u(t-1))^2}{8sin^4(\frac{\pi}{2N})} + sin^2(\frac{\pi t}{N}),\ t\in I;\\ u(0)=0, u(N)=0;\\ u(-1)+u(1)=0, u(N-1)+u(N+1)=0.\end{array}\right\} \quad (4.1)$$

We have,

$$f(t,u(t),\Delta^2 u(t-1)) = u^2(t) - \frac{(\Delta^2 u(t-1))^2}{8sin^4(\frac{\pi}{2N})} + sin^2\left(\frac{\pi t}{N}\right).$$

Let $\alpha(t) = 0$, $\beta(t) = sin(\frac{\pi t}{N})$. It is obvious that $\alpha(t)$ is a lower solution of (4.1). We have

$$\begin{aligned}\Delta^2\beta(t-1)) &= -4sin\left(\frac{\pi t}{N}\right) sin^2\left(\frac{\pi}{2N}\right).\\ \Delta^4\beta(t-2)) &= 16sin^4\left(\frac{\pi}{2N}\right) sin\left(\frac{\pi t}{N}\right)\\ f(t,\beta(t),\Delta^2\beta(t-1)) &= 0.\end{aligned}$$

Therefore we have

$$\begin{aligned} \Delta^4\beta(t-2) &\geq f(t,\beta(t),\Delta^2\beta(t-1)); \\ \beta(0)=0, &\quad \beta(N)=0; \\ \beta(-1)+\beta(1)=0, &\quad \beta(N-1)+\beta(N+1)=0. \end{aligned}$$

Hence $\beta(t)$ is an upper solution of (4.1). Moreover, $\alpha(t) \leq \beta(t)$ for $t \in J$ and

$$\Delta^2\alpha(t-1) \geq \Delta^2\beta(t-1)), \ t \in I.$$

Thus (H_1) holds. Moreover for any $v \in E$

$$f(t,u(t),v(t)) = u^2(t) - \frac{v^2(t)}{8sin^4(\frac{\pi}{2N})} + sin^2\left(\frac{\pi t}{N}\right).$$

For any $u_1, u_2 \in E$ with $\alpha(t) \leq u_1(t) \leq u_2(t) \leq \beta(t)$,

$$u_1^2 \leq u_2^2 \Rightarrow f(t,u_2(t),v(t)) - f(t,u_1(t),v(t)) \geq 0.$$

Thus (H_2) holds. Lastly, for any u, v_1, v_2 satisfying

$$0 \leq u(t) \leq sin\left(\frac{\pi t}{N}\right) and - 4sin\left(\frac{\pi t}{N}\right) sin^2\left(\frac{\pi}{2N}\right) \leq v_1 \leq v_2 \leq 0,$$

we have

$$f(t,u,v_2) - f(t,u,v_1) \leq M(v_2 - v_1),$$

where $M = \frac{1}{sin^2 \frac{\pi}{2N}} > 0.$ Thus (H_3) also holds. Hence by Theorem 3.1, the BVP (4.1) has external solutions in $[\alpha(t), \beta(t)]$.

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