Special Cases in a Mathematical Model of an Ecological Ammensalism with Unlimited Resources – A Numerical Study

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Abstract

The paper purports to examine the nature and the interactions between the species in some special cases of a mathematical model of an ecological Ammensal model in which Ammensal (S_1) and enemy (S_2) species are surviving with unlimited resources. The model is formed by a coupled system of first order non-linear ordinary differential equations. The only one equilibrium point is noticed and the criteria for its stability are inferred. Solutions for the linearised perturbed equations are determined. Further the numerical solutions for this model are computed for identifying the nature and the interactions between the species by using Runge-Kutta method of fourth order.

Keywords: Equilibrium points, Normal steady state, stability, R.K method of fourth order.

Mathematics Subject Classification: **92D25, 92D40**

1. Introduction

A mathematical model represents a form of a system of rules using mathematical language. The procedure of formulating and analyzing a model is named as mathematical modeling. Because of its scientific significances,it is used not only in the field natural Sciences such as Physics, Mycology, Earth science, Meteorology but also in the area of engineering disciplines such as Computer science, Artificial intelligence. Mathematical models are most extensively used in dynamical systems, statistical models, differential equations, or game theoretic models. Mathematical modeling in theoretical ecology was pioneered by Cushing [1] and Meyer [12] espoused by several

mathematicians and ecologists .They added their findings to the growth of this area of knowledge as reported in the treatises of Kapur [2]. Srinivas [11] studied competitive eco-systems of two and three species with limited and unlimited resources. Acharyulu and Pattabhi Ramacharyulu [3-10] investigated some results on stability of an enemy and Ammensal species pair with various resources.

The present paper handles the formation and analytical/numerical investigation on a Mathematical model of an Ammensal (S_1) and enemy (S_2) species pair with unlimited resources paying care on the criteria for stability of the equilibrium states. Diverse mathematical and numerical techniques such as linearization of the basic equations, Runge-Kutta method of Fourth order and genetic algorithm are employed to examine this model.

1.1. Notation adopted

N_1, N_2: The populations of the Ammensal (S_1) and enemy (S_2) species respectively at time t.

a_1, a_2 : The natural growth rates of S_1 and S_2

a_{12} : The Ammensal coefficient.

Further both the variables N_1 and N_2 are non-negative and the model parameters a_1, a_2, and a_{12} are assumed to be non-negative constants. Employing the above terminology, the model equations for a two species Ammensal system are constructed as below.

2. Basic Equations

The equation for the growth rate of Ammensal species (S_1) is given by

$$\frac{dN_1}{dt} = a_1 N_1 - a_{12} N_1 N_2 \quad (1)$$

The equation for the growth rate of enemy species (S_2) is given by

$$\frac{dN_2}{dt} = a_2 N_2 \quad (2)$$

The equilibrium states are obtained by $\frac{dN_1}{dt}=0$ and $\frac{dN_2}{dt}=0$.

That is $N_1\{a_1 - a_{12} N_2\} = 0$ and $a_2 N_2 = 0$ (3)

A solution $(\overline{N_1}, \overline{N_2})$ of (3) is called the equilibrium state of (1) and (2)

The system under this investigation has only one equilibrium state

given by $\overline{N_1}=0$; $\overline{N_2}=0$ (4)

In this state both the species are washed out.

The basic equations (1) and (2) are linearised to obtain the system for the perturbed state,

$$\frac{dU}{dt}=AU \tag{5}$$

$$\text{Where } \begin{bmatrix} a_1-a_{12}\overline{N_2} & -a_{12}\overline{N_1} \\ 0 & a_2 \end{bmatrix} \tag{6}$$

The characteristic equation for the system is Det $[A-\lambda I]=0$ (7)

The equilibrium state is stable, if both the roots of the equation (7) are negative.

In order to discuss the stability of equilibrium state $\overline{N_1}=0$; $\overline{N_2}=0$,

we have to consider small perturbations $U_1(t)$ and $U_2(t)$ from the steady state.

That is $N_1=\overline{N_1}+U_1(t)$, $N_2=\overline{N_2}+U_2(t)$ (8)

where U_1 (t) and U_2 (t) are so small that the terms other than their first terms can be neglected. Substituting (8) in (1) and (2), and neglecting products and higher powers of U_1 and U_2 we get

$$\frac{dU_1}{dt} = a_1 U_1 \text{ and } \frac{dU_2}{dt} = a_2 U_2 \tag{9}$$

the characteristic Equation for which is $(\lambda-a_1)(\lambda-a_2)=0$, the roots a_1, a_2 of which are both positive. Hence the steady state is **unstable**.

Solving (9), we get $U_1=U_{10}\, e^{a_1 t}$ and $U_2=U_{20}\, e^{a_2 t}$ (10)

where U_{10}, U_{20} are initial values of U_1, U_2 respectively.

The above solutions establish the fact that both the species increase unboundedly and monotonically.

The solution curves are illustrated in figures from Figure-1 to Figure-4.

Case-1: $a_1>a_2$ **and** $U_{10}>U_{20}$

i.e the Ammensal (S_1) dominates over the enemy (S_2) in natural growth rate as well as its initial population strength.Clearly the Ammensal(S_1) species continues to be out numbering the enemy(S_2) species as shown in Figure (1).

2.1. Trajectories of Perturbed Species

The trajectories (solution curves of 10) in the $U_1 - U_2$ plane are given by

$$\left(\frac{U_1}{U_{10}}\right)^{a_2} = \left(\frac{U_2}{U_{20}}\right)^{a_1}$$ and these are illustrated in Figure (5)

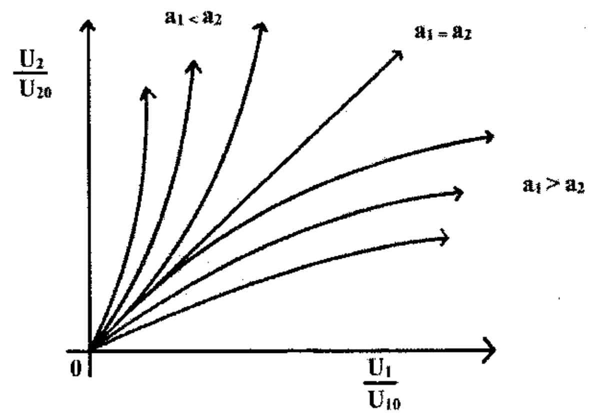


Figure (5) : Trajectories of Perturbed Species

3. The solutions of the model obtained by the classical Runge-kutta method of fourth order with the help of Genetic Alogorithm:

The exiting cases in this model are classified into two cases.

Case(A): $a_1 > a_2$ and $N_{10} > N_{20}$

Case(B): $a_1 < a_2$ and $N_{10} > N_{20}$

We have traced out all most all possible solutions in the finite interval by employing the classical RK method of fourth order with the aid of Genetic Algorithm. The interval is assumed to range over 0 to 5 for observing the nature of the model. The gained solutions are framed in the tabular forms and the graphs are illustrated wherever necessary.

Case(A): The natural growth rate of Ammensal species is greater than the natural growth rate of enemy species in which the initial strength of the Ammensal species is equal to the initial strength of enemy species i.e $a_1 > a_2$ and $N_{10} > N_{20}$ .The obtained solutions are given in the Table-1

Table-1

S.NO	a_1	a_{12}	a_2	N_{10}	N_{20}	t^*
1	4.474076	1.989289	2.890029	4.807993	4.641643	0.004
2	1.959451	1.387418	0.429724	4.802141	4.668355	0.005
3	1.959451	1.387418	0.637817	4.802141	4.644861	0.006
4	3.971597	1.961564	2.911426	3.714524	3.557857	0.007
5	3.971597	1.961564	2.911426	3.714524	3.557857	0.007
6	1.456972	1.359693	0.451121	3.708671	3.584569	0.008
7	3.971597	1.961564	2.888664	3.770582	3.557857	0.009
8	2.041296	1.228625	0.429724	4.940283	4.668355	0.013
9	4.417756	3.692808	2.438319	3.587861	2.992665	0.019
10	3.451983	4.334288	2.557725	2.06624	1.773655	0.021
11	4.185469	3.159696	1.629997	3.346744	2.627418	0.039
12	4.067874	3.114287	3.925245	3.346744	2.433893	0.039
13	4.497	4.652572	3.039781	4.000926	2.546156	0.04
14	4.067874	3.159696	3.21174	3.346744	2.433893	0.043
15	4.066058	3.159696	3.21174	3.385775	2.433893	0.044
16	2.854588	2.422934	2.446668	3.346744	2.433893	0.053
17	1.959451	1.361059	0.429724	4.802141	3.827494	0.059
18	4.079527	1.378359	2.794691	3.799637	3.024302	0.069
19	3.870991	0.409059	2.891591	3.751985	3.570254	0.076
20	4.161373	1.219566	2.794691	3.81791	3.01603	0.084
21	4.363967	1.638284	0.768296	4.994785	3.803581	0.094
22	2.63732	4.69976	0.820299	2.255507	1.388942	0.096
23	4.067874	1.002073	3.21174	3.346744	2.433893	0.142
24	4.067874	3.159696	1.484701	3.304396	1.782176	0.162
25	4.637999	2.071257	3.306299	3.771998	1.411622	0.281
26	3.235197	1.63496	1.592739	3.300101	1.773829	0.299
27	3.358309	1.618106	1.592739	3.30158	1.773829	0.315

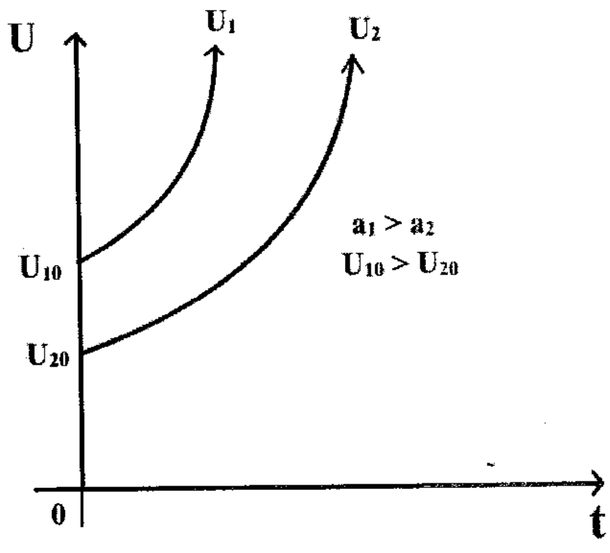


Figure (1) : Both the species flourishing with the condition $a_1 > a_2$ and $U_{10} > U_{20}$

Case-2 : $a_1 > a_2$ **and** $U_{10} < U_{20}$

i.e. the Ammensal (S_1) species dominates the enemy (S_2) species in the natural growth rate but its initial strength is less than that of the enemy species. Here the Ammensal (S_1) out - numbers the enemy(S_2) till the time - instant $t^* = \frac{1}{a_2 - a_1} \log\left(\frac{U_{10}}{U_{20}}\right)$ after that the dominance is reversed. This is depicted in Figure (2).

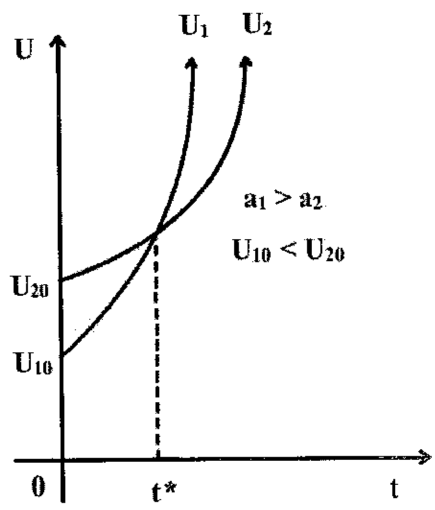


Figure (2) : Both the species flourishing with the condition $a_1 > a_2$ **and** $U_{10} < U_{20}$

Case-3: $a_1 < a_2$ **and** $U_{10} > U_{20}$ i.e. the enemy species dominates the Ammensal species in the natural growth rate but its initial strength is less than that of Ammensal species.

In this case, the Ammensal (S_1) species out -numbers the enemy (S_2) species till the time-instant,

$t = t^* = \frac{1}{a_2 - a_1}\log\left(\frac{U_{10}}{U_{20}}\right)$ After that the enemy (S_2) species outnumbers the Ammensal (S_1) species. This is illustrated in Figure (3).

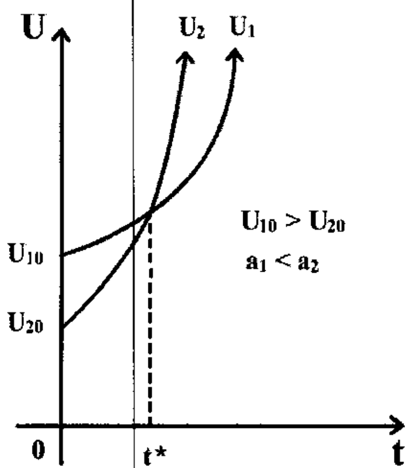


Figure (3) : Both the species flourishing with the condition $a_1 < a_2$ and $U_{10} > U_{20}$

Case-4: $a_1 < a_2$ **and** $U_{10} < U_{20}$

That is the enemy species (S_2) dominates the Ammensal(S_1) species in the natural growth rate as well as in its initial population strength. In this case, the enemy (S_2) species continues to dominate the Ammensal (S_1) species as shown in Figure (4).

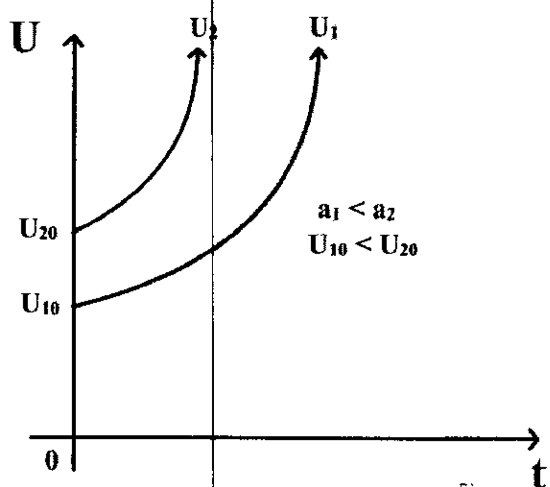


Figure (4): Both the species flourishing with the condition $a_1 < a_2$ **and** $U_{10} < U_{20}$

28	3.358309	1.618106	1.592739	3.30158	1.773829	0.315
29	3.358309	1.618106	1.592739	3.30158	1.773829	0.315
30	3.358309	1.618106	1.592739	3.30158	1.773829	0.315
31	2.939548	1.992347	1.365354	3.790382	1.537522	0.372
32	1.737994	1.992347	0.767142	3.757125	1.475342	0.381
33	4.407108	1.618106	1.592739	3.30158	1.773829	0.449
34	2.878557	2.283784	2.588399	4.228792	0.866601	0.455
35	2.939548	1.992347	0.913313	3.796565	1.537522	0.49
36	2.939548	1.992347	0.913313	3.796565	1.537522	0.49
37	2.939548	1.992347	0.913313	4.349267	1.537522	0.537
38	2.939548	1.992347	0.784219	3.796565	1.53244	0.546
39	2.998374	3.524386	0.718184	1.889354	0.856145	0.551
40	2.939548	1.992347	0.767142	3.749651	1.475342	0.597
41	3.478751	1.992347	0.913313	3.796565	1.537522	0.597
42	3.358309	1.618106	0.759804	3.30158	1.773829	0.598
43	3.16453	1.992347	0.711764	3.644346	1.396646	0.766
44	2.995035	1.948832	2.000935	2.568676	0.591897	0.796
45	2.998374	3.524386	0.457394	2.401495	0.854131	0.88
46	2.998374	3.524386	0.457394	2.401495	0.854131	0.88
47	3.95509	1.991799	0.864644	3.161425	1.304753	0.947
48	4.637999	0.747338	3.306299	1.123338	0.362129	1.047
49	4.637999	0.747338	3.306299	1.123338	0.362129	1.047
50	1.116665	1.355364	0.457394	2.401495	0.854131	1.176
51	2.180698	0.858433	1.53444	1.663381	0.514345	1.367
52	2.613487	2.093461	0.667318	2.568676	0.685863	1.734

Some of the picked out solution curves are illustrated in figures from Figure (6) to Figure (19)

Figure (6) ; S.N0-1 in TABLE-1

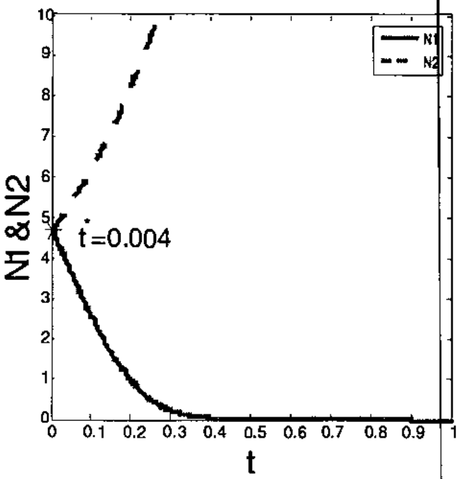


Figure (7) ; S.N0-4 in TABLE-1

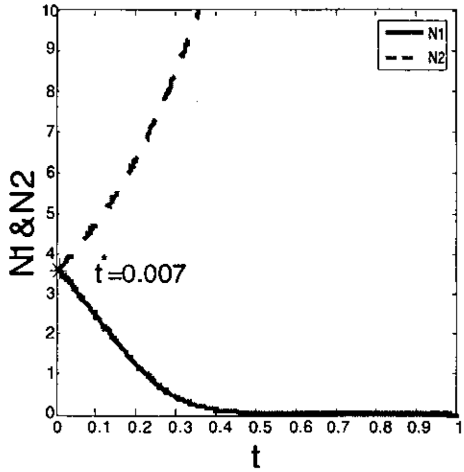


Figure (8) ; S.N0-8 in TABLE-1

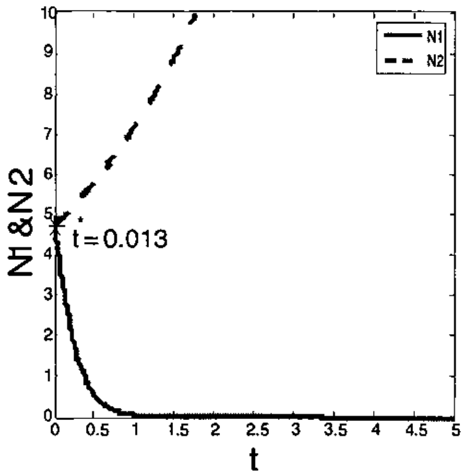


Figure (9) ; S.N0-12 in TABLE-1

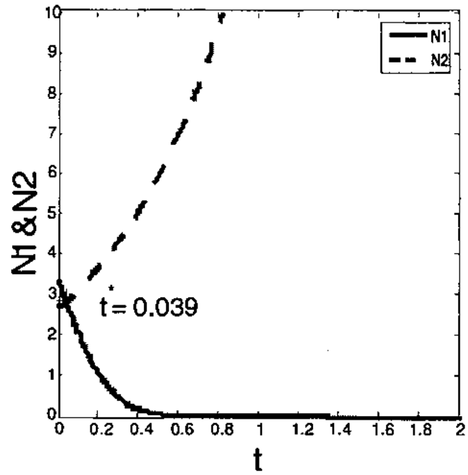


Figure (10) ; S.N0-16 in TABLE-1

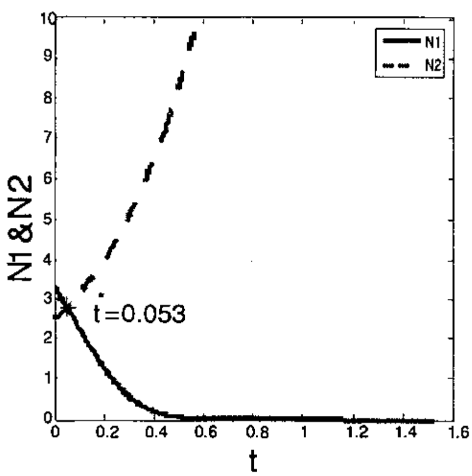
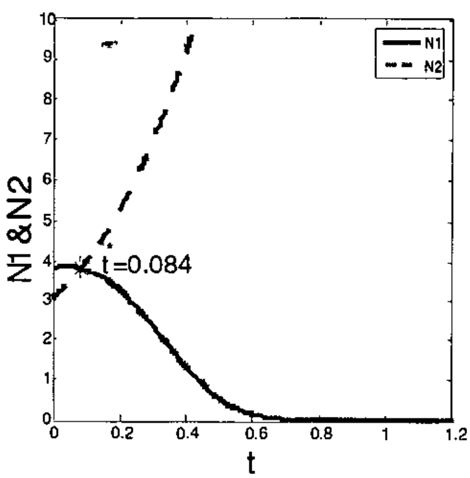


Figure (11) ; S.N0-20 in TABLE-1



Figure(12) ; S.N0-24 in TABLE-1

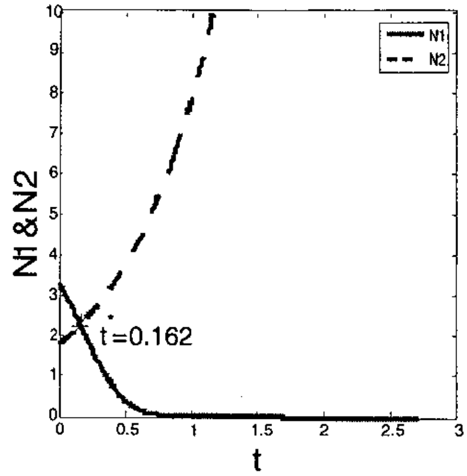


Figure (13) ; S.N0-28 in TABLE-1

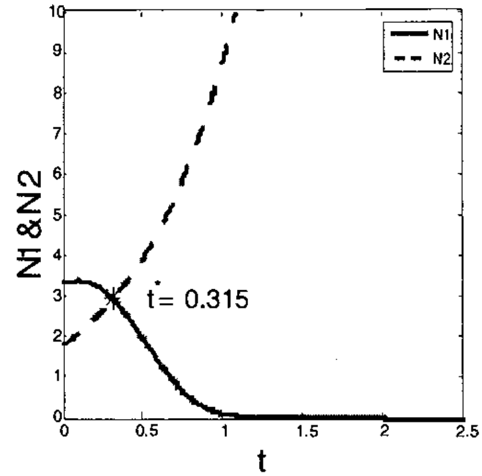


Figure (14) ; S.N0-32 in TABLE-1

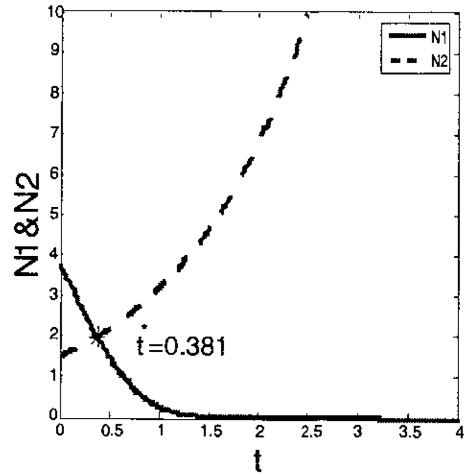


Figure (15) ; S.N0-36 in TABLE-1

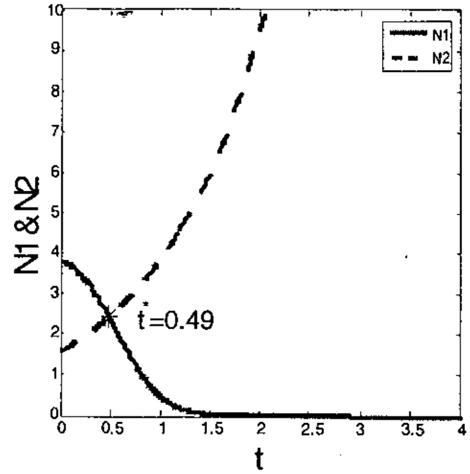


Figure (16) ; S.N0-40 in TABLE-1

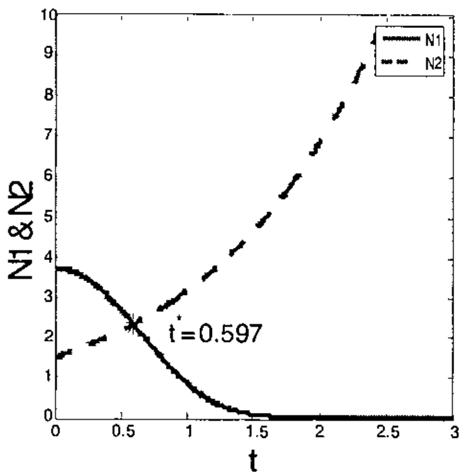


Figure (17) ; S.N0-44 in TABLE-1

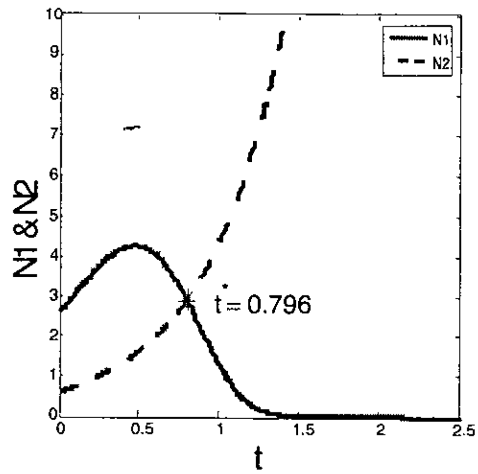


Figure (18) ; S.N0-48 in TABLE-1

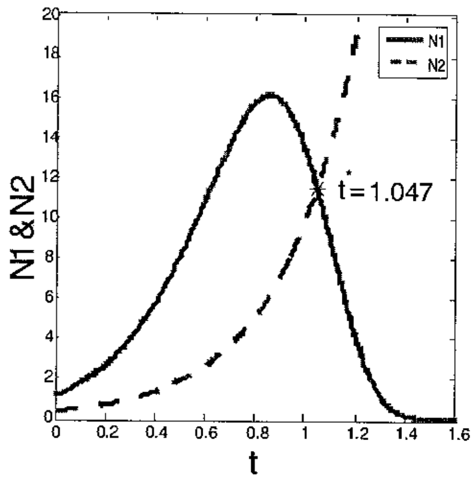
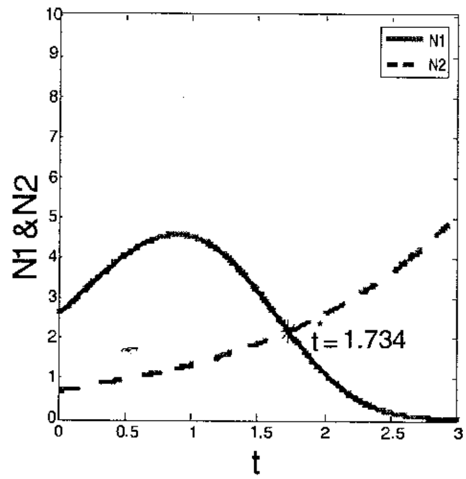


Figure (19) ; S.N0-52 in TABLE-1



Figure(6) to (19): Various Solution curves selected with dominance- reversal time(t*) along with the conditions $a_1 > a_2$ and $N_{10} > N_{20}$ in the considered interval.

Conclusion

In this case the Ammesal (S_1) dominates over the Enemy (S_2) in natural growth as well as in its initial population.The Ammensal outnumbers the enemy till the dominance- reversal time t* after that the enemy outnumbers the Ammensal in the considered interval. However the Ammensal prospers in growth rate up to t* and then decays in the remaining interval .It is evident that the enemy species flourihes through out the interval. (11)

4. Case(B) : The natural growth rate of Ammensal species is less than the natural growth rate of enemy species in which the initial strength of the Ammensal species is greater than the initial strength of enemy species i.e $a_1 < a_2$ and $N_{10} > N_{20}$. The obtained solutions are given in the Table-2.We have traced out all most all possible solutions in the finite interval by employing the classical RK method of fourth order with the aid of Genetic Algorithm. The interval is assumed to range over 0 to 5 for observing the nature of the model.The gained solutions are framed in the tabular form and the graphs are illustrated wherever necessary.

Table 2

S.NO	a_1	a_{12}	a_2	N_{10}	N_{20}	t
1	4.194843	3.800122	4.368445	4.03438	3.866699	0.002162
2	1.408383	1.360301	4.431313	4.03438	3.866699	0.003394
3	4.333598	1.197372	4.605126	2.592664	2.549753	0.003844
4	1.045045	4.652572	2.028459	2.719874	2.546156	0.004382
5	3.409809	1.961564	4.712237	3.829394	3.553849	0.004555
6	0.15572	4.795719	4.426985	4.2816	3.607222	0.007297
7	0.098408	4.795719	4.426985	4.2816	3.607222	0.007297
8	0.670457	3.416104	2.029818	3.068893	2.63264	0.011095
9	0.15572	1.337246	4.426985	4.2816	3.787453	0.012509
10	0.640771	3.123347	2.029818	3.002506	2.63264	0.012967
11	4.194843	3.685881	4.431313	4.864094	3.866699	0.013575
12	0.327635	2.418222	2.667253	4.290689	3.607222	0.015115
13	3.697567	2.315663	4.235985	2.5427	2.266294	0.01768
14	1.954229	0.453198	2.532172	3.714524	3.557857	0.017866
15	0.103189	4.969631	3.05903	3.721961	2.600407	0.021406
16	4.073802	0.19005	4.941244	1.894774	1.827462	0.022909
17	1.378315	2.298181	2.98548	3.103364	2.546156	0.024829
18	1.378315	4.652572	3.177017	4.04784	2.608336	0.029697
19	1.378315	4.829977	3.030846	4.000926	2.546156	0.030671
20	1.378315	4.815543	2.837055	4.000926	2.546156	0.030847
21	1.378315	4.652572	3.030846	4.000926	2.546156	0.031644
22	1.378315	4.652572	3.030846	4.000926	2.546156	0.031644
23	1.378315	4.652572	3.030846	4.000926	2.546156	0.031644
24	1.378315	4.652572	2.578805	4.007109	2.546156	0.033105
25	1.631296	1.258022	2.438319	3.587861	2.992665	0.03687
26	1.042687	1.219566	2.785755	3.799637	3.024302	0.039557
27	1.058979	2.891585	3.030846	4.000926	2.546156	0.045428
28	4.253618	1.219566	4.716594	3.820122	3.024302	0.048807
29	0.670457	0.662197	1.988278	3.211203	2.63264	0.059175

30	1.378315	4.652572	3.030846	4.000926	1.92787	0.062802
31	1.981144	1.735133	2.438319	3.587861	2.395889	0.078219
32	3.364738	0.166815	4.235985	2.5427	2.21135	0.102294
33	3.614674	0.166815	4.235985	2.5427	2.266294	0.102294
34	0.180357	2.193553	1.573017	3.30158	1.558722	0.140446
35	3.954055	2.25254	4.714925	3.095449	1.304753	0.164532
36	3.364738	0.166815	4.235985	3.130349	2.260426	0.217216
37	1.378315	4.652572	2.284491	4.000926	0.844185	0.249746
38	1.840975	2.283784	2.396827	4.333561	1.283341	0.259137
39	0.473309	2.418222	2.692541	4.290689	0.262963	0.686323
40	1.681888	1.336859	2.014261	2.424692	0.589065	0.690254
41	1.681888	1.336859	2.014261	2.424692	0.589065	0.690254
42	3.836897	3.875931	4.364029	0.929154	0.057195	0.880196

Some of the recognized solution curves are illustrated from Figure(20) to Figure(29)

Figure (20) ; S.N0-1 in TABLE-2

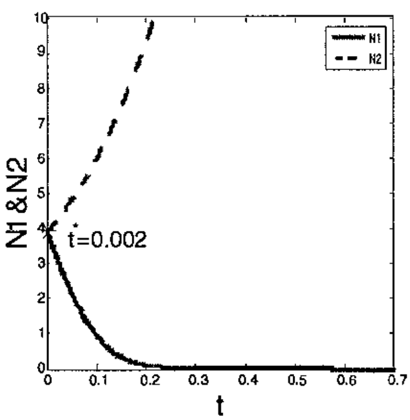


Figure (21) ; S.N0-5 in TABLE-2

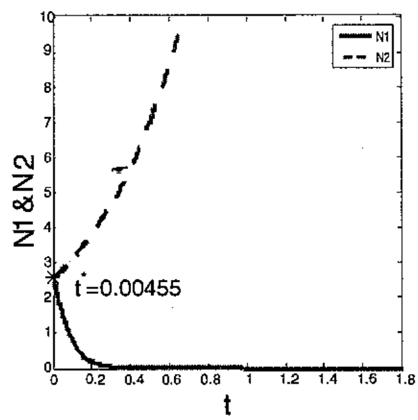
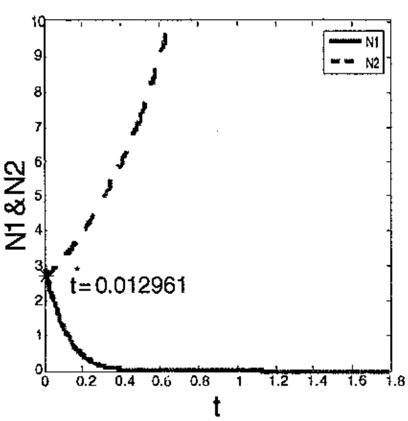


Figure (22) ; S.N0-10 in TABLE-2



Figure(23) ; S.N0-15 in TABLE-2

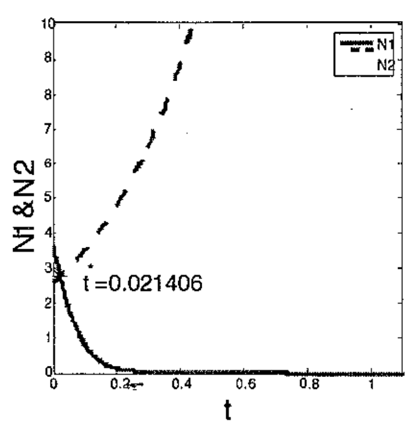


Figure (24) ; S.N0-20 in TABLE-2

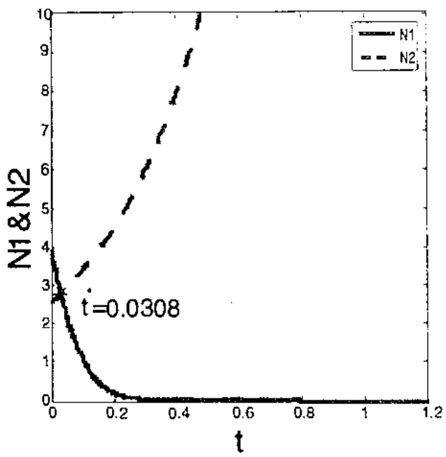


Figure (25) ; S.N0-25 in TABLE-2

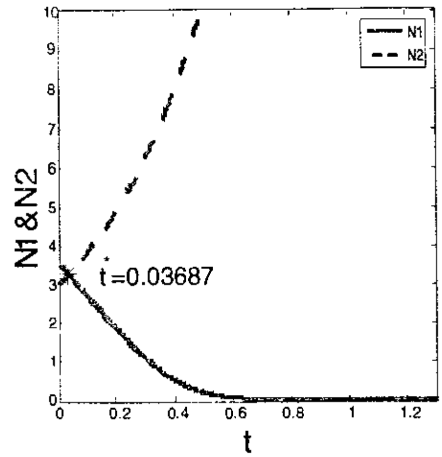


Figure (26) ; S.N0-30 in TABLE-2

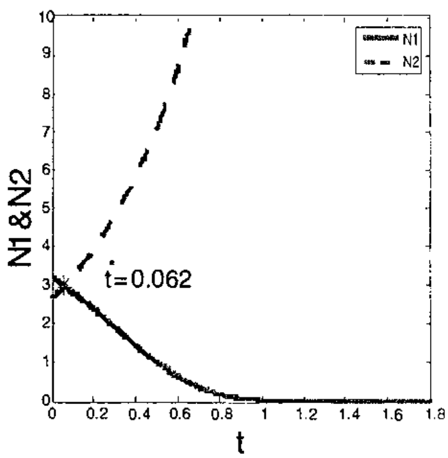


Figure (27) ; S.N0-35 in TABLE-2

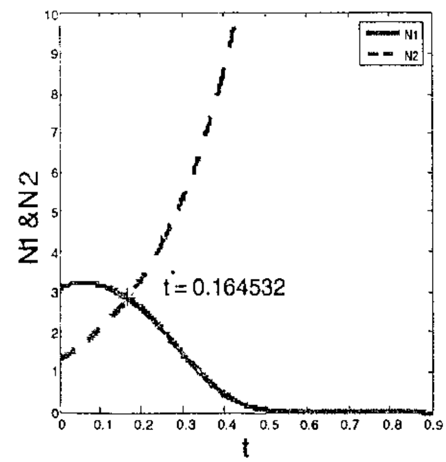
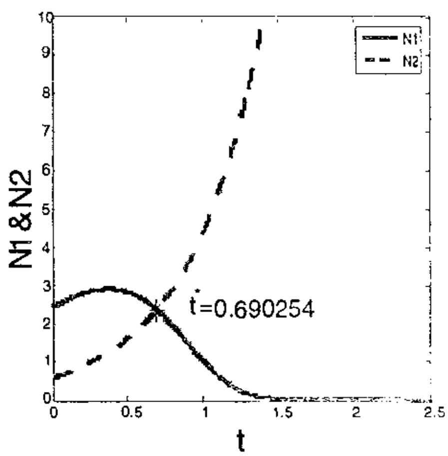
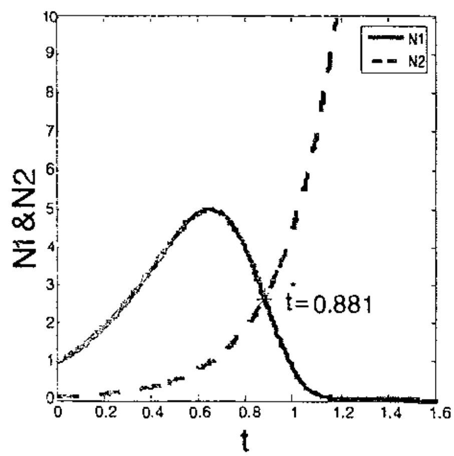


Figure (28) ; S.N0-40 in TABLE-2



Figure(29) ; S.N0-42 in TABLE-2



Figure(20) to (29): Various Solution curves picked up with dominance- reversal time(t*) along with the conditions $a_1 < a_2$ and $N_{10} > N_{20}$ in the specified interval.

5. Conclusion

The enemy dominates over the Ammensal species in natural growth where as Ammensal eclipses enemy in its initial population strength. More over Ammensal surpasses the enemy till the dominance- reversal time t* after that the enemy surpasses the Ammensal in the considered interval. Further it is noticed that there is a steep down in the Ammensal spieces and the enemy persists gradually with exponential growth.

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