Congruence Relation and Ternary Operation on Pre A*-Algebra

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ABSTRACT

This paper is a study on algebraic structure of Pre A*- algebra. We prove that the relation on Pre A*-algebra, $\theta_a = \{(x,y) \in A \times A \,/\, a \wedge x = a \wedge y\}$ is a congruence relation. We define the ternary operation on Pre A*-algebra as $\Gamma_x(p,q) = (x \wedge p) \vee (x^{\sim} \wedge q)$, ($\Gamma_x(p,q)$ should be viewed as conditional 'if x, then p, else q') and studied its properties. We define $\psi_x = \{(p,q) \in A \times A \,/\, \Gamma_x(p,q) = p\}$ on Pre A*-algebra and we find the necessary and sufficient conditions for the above relation to be an equivalence relation. We define factor congruence on A and prove that if $\theta = \theta_x$ for some $x \in B(A)$, then θ_x is a factor congruence on A.

Keywords: Algebraic structure; Pre A*- algebra; Congruence relation; Ternary operation on Pre A*-algebra; Equivalence relation; Factor congruence relation.

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1. INTRODUCTION

In a draft paper Manes E.G., (1989) introduced the concept of Ada $(A, \wedge, \vee, (-)', (-)_\pi, 0, 1, 2)$, (Algebra of Disjoint Alternatives) which however differs from the definition of the Ada of Manes E.G. (1993) of his later paper. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras, the later concept is based on C-algebras $(A, \wedge, \vee, (-)^{\sim})$ introduced by Fernando Guzman and Craig C. Squir (1990).

P.Koteswara Rao (1994) first introduced the concept of A*-algebra $(A, \wedge, \vee, *, (-)^{\sim} (-)_\pi, 0, 1, 2)$ and studied the equivalence with Ada of Manes E.G (1993), C-algebra of Fernando Guzman and Craig C. Squir (1990), and Ada of Manes E.G (1993) and its connection with 3-ring, Stone type representation and introduced the concept of A*-clone and the If-Then-Else structure over A*-algebra and ideal of A*-algebra. J.Venkateswara Rao (2000) introduced the concept Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$ analogous to C-algebra as a reduct of A*- algebra. Venkateswara Rao. J and Srinivasa Rao. K, (2009) introduced the Congruence relations on Pre A*-Algebra. Further Venkateswara Rao. J and Srinivasa Rao. K, (2010) obtained the well known Cayley's theorem on centre of Pre A*-algebras. Also recently Venkateswara Rao.J, Srinivasa Rao, K., and Satyanarayana, A, (2010) obtained some Structural Compatibilities of Pre A*-Algebra.

1. PRELIMINARIES:

In this section we concentrate on the algebraic structure of Pre A*-algebra and prove some results which will be used in the later text.

1.1. Definition:

An algebra $(A, \wedge, \vee, (-)^{\sim})$ satisfying

(a) $x^{\sim\sim}=x, \forall x \in A$, (b) $x \wedge x = x, \forall x \in A$, (c) $x \wedge y = y \wedge x, \forall x, y \in A$,

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x, y \in A$, (e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$,

(f) $x \wedge (y \vee z) = (x\ y) \vee (x \wedge z), \forall x, y, z \in A$, (g) $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y, z \in A$

is called a Pre A*-algebra.

1.2. Example:

3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

1.3. Note : The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim}=2$ (b) $1 \wedge x = x$ for all $x \in 3$ (c) $0 \vee x = x$ for all $x \in 3$

(d) $2 \wedge x = 2 \vee x = 2$ for all $x \in 3$.

1.4. Example :

2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

1.5. Note:

(i) $\left(2, \vee, \wedge, (\tilde{-})\right)$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra

(ii) The identities 1.1(a) and 1.1(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.1(a) to 1.1(g).

1.6. Lemma :

Every Pre A*-algebra with 1 satisfies the following laws

(a) $x \vee 1 = x \vee x^{\sim}$ (b) $x \wedge 0 = x \wedge x^{\sim}$

Proof:

(i) $x \vee 1 = x \vee (x^{\sim} \wedge 1)$ (By dual of 1.1(g))

$= x \vee x^{\sim}$ (since $x \wedge 1 = x$, $\forall x \in A$)

(i) $x \wedge 0 = x \wedge (x^{\sim} \vee 0)$ (By 1.1(g))

$= x \vee x^{\sim}$ (since $x \vee 0 = x$, $\forall x \in A$)

1.7. Lemma :

Every Pre A*-algebra satisfies the following laws.

(a) $x \vee (x^{\sim} \wedge x) = x$ (b) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

(c) $(x \vee x^{\sim}) \wedge x = x$ (d) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

Proof:

(a) $x \wedge y = x \wedge (x^{\sim} \vee y)$ (By 1.1 (g)) $\Rightarrow x \wedge x = x \wedge (x^{\sim} \vee x)$

$\Rightarrow (x \wedge x)^{\sim} = (x \wedge (x^{\sim} \vee x))^{\sim} \Rightarrow x^{\sim} = x^{\sim} \vee (x^{\sim} \vee x)^{\sim}$ (By 1.1 (b))

$\Rightarrow x^{\sim} = x^{\sim} \vee (x \wedge x^{\sim}) \Rightarrow x = x \vee (x^{\sim} \wedge x)$

(b) Use 1.1(f) and 1.1(e)

(c) $(x \vee x^{\sim}) \wedge x = (x \wedge x) \vee (x^{\sim} \wedge x)$ (By 1.7 (b))

$= x \vee (x^{\sim} \wedge x) = x$ (By 1.1 (b), 1.7(a))

(d) $(x \wedge z) \vee (x^{\sim} \wedge y \wedge z) = (x \vee (x^{\sim} \wedge y)) \wedge z = (x \wedge (x^{\sim} \vee y)) \wedge z = (x \vee y) \wedge z$

1.8. Definition :

Let A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x \vee x^{\sim}=1$ and the set $\{x \in A / x \vee x^{\sim}=1\}$ of all central elements of A is called the centre of A and it is denoted by B (A).

Note that if A is a Pre A*-algebra with 1 then $1, 0 \in B(A)$. If the centre of Pre A*-algebra coincides with {0, 1} then we say that A has trivial centre.

1.9. Theorem :

Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

Proof:

First we prove that B (A) is a sub algebra of A.

Suppose $x, y \in B(A)$ then $x \vee x^{\sim} = y \vee y^{\sim} = 1$. If $x \vee x^{\sim} = 1$ *then* $x^{\sim} \vee (x^{\sim})^{\sim} = 1 \Rightarrow x^{\sim} \in B(A)$ $(x \vee y) \vee (x \vee y)^{\sim} = (x \vee y) \vee 1$ (By 1.6(a)) $= x \vee (y \vee 1) =$ $x \vee (y \vee y^{\sim})$ (By 1.6(a)) $= x \vee 1 = x \vee x^{\sim} = 1$ (By 1.6(a))

Therefore $x \vee y \in B(A)$ and hence B (A) is sub algebra of A.

Also since $1 \vee 0 = 1$, $1 \in B(A)$ and each $x \in B(A)$, $x \wedge 1 = x \wedge (x \vee x^{\sim}) = x$ (By 1.7(c)),

hence $1 (= x \vee x^{\sim})$, which is an identity for $\wedge$ in B (A). Then by lemma 1.7(a), $x \vee (x^{\sim} \wedge x) = x \Rightarrow x \vee 0 = x$ (since, $x \in B(A)$ $x \vee x^{\sim} = 1 \Rightarrow x \wedge x^{\sim} = 0$

This shows that $0 (= x \wedge x^{\sim})$. is an identify for $\vee$ in B(A)

Now we prove that $x \vee y^{\sim} = 1$ if and only if $x \vee y = x$

Suppose that $x \vee y^{\sim} = 1$

By dual of 1.1(g), we have $x \vee y = x \vee (x^{\sim} \wedge y) = x \vee 0 = x$ (since $x \vee y^{\sim} = 1 \Rightarrow x^{\sim} \wedge y = 0$)

Suppose that . $x \vee y = x$ Then $x \vee y^{\sim} = x \vee y \vee y^{\sim} = x \vee 1 = 1$

Hence B (A) is a Boolean algebra.

1.10. Lemma :

Let A be a Pre A*-algebra with 1.

(a) If $x, y \in B(A)$ then $x \wedge x^{\sim} \wedge y = x \wedge x^{\sim}$

(b) $x \wedge (x \vee y) = x \vee (x \wedge y) = x$ if and only if $x, y \in B(A)$

Proof:

(a) If $x, y \in B(A)$, then $x \vee x^{\sim} = 1$ $(x \wedge x^{\sim} = 0)$ and $y \vee y^{\sim} = 1$ $(y \wedge y^{\sim} = 0)$

Now $x \wedge x^{\sim} \wedge y = 0 \wedge y = 0$ (since $y \in B(A)$) $= x \wedge x^{\sim}$

(b) Suppose that $x, y \in B(A)$, $x \wedge (x \vee y) = (x \wedge x) \vee (x \wedge y) = x \vee (x \wedge y)$

$$= x \vee (x \wedge (x^{\sim} \vee y)) \text{ (By 1.1 (g))} = (x \vee x) \wedge (x \vee (x^{\sim} \vee y)) \text{ (By dual of (a))}$$

$$= x \wedge (x \vee x^{\sim}) = x \quad \text{(By 1.7(a))}$$

Suppose that $x \wedge (x \vee y) = x \vee (x \wedge y) = x$

1.11. Note :

By 1.3 we have, if $a \in A$ such that $a^{\sim} = a$ then $a \wedge x = a \vee x = a$, $\forall x \in A$, in this case $x \wedge (x \vee a) = x \vee (x \wedge a) = a$ and $a \wedge a^{\sim} \wedge x = a \wedge a^{\sim}$, $\forall x \in A$

1.12. Lemma :

Let A be a Pre A*algebra with 1, 0 and let x, y $\in$A

(a) If $x \vee y = 0$, then x = y = 0 (b) If $x \vee y = 1$, then $x \vee x^{\sim} = 1$

Proof:

(a) $x = x \vee 0 = x \vee x \vee y (\because x \vee y = 0) = x \vee y = 0$(Dual of 1.1b)

Similarly we can prove that y = 0.

(b) $1 = x \vee y = x \vee (x^{\sim} \wedge y)$ (Dual of 1.1 g)= $(x \vee x^{\sim}) \wedge (x \vee y) = (x \vee x^{\sim}) \wedge 1 = (x \vee x^{\sim})$

1.13. Theorem :

Let A be a Pre A*-algebra with 1 and $x, y \in A$, if $x \wedge y = 0$, $x \vee y = 1$, then $y = x^{\sim}$.

Proof:

If $x \vee y = 1$, then $x \vee x^{\sim} = 1$ (By 1.12(b)) $\Rightarrow x^{\sim} \wedge x = 0$

Now $y = 1 \wedge y = (x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

$= 0 \vee (x^{\sim} \wedge y) = (x^{\sim} \wedge x) \vee (x^{\sim} \wedge y) = x^{\sim} \wedge (x \vee y) = x^{\sim} \wedge 1 = x^{\sim}$

2. CONGRUENCE RELATION ON PRE A*-ALGEBRA

In this section we prove that the relation $\theta_a = \{(x, y) \in A \times A /\ a \wedge x = a \wedge y\}$ is a congruence relation on Pre A*-algebra and we prove that the set A/θ = $\{\theta_a / a \in A\}$ is a Pre A*-algebra.

2.1. Definition :

A relation θ on a Pre –A* algebra (A, ∧, ∨, $(-)^{\sim}$) is called congruence relation if (i) is an equivalence relation (ii) θ is closed under ∧, ∨, $(-)^{\sim}$.

2.2. Lemma :

Let (A, ∧, ∨, $(-)^{\sim}$) be a Pre A*-algebra and let $a \in A$. Then the relation

$\theta_a = \{(x, y) \in A \times A /\ a \wedge x = a \wedge y\}$ is (a) θ_a is a congruence relation

(b) $\theta_a \cap \theta_{a^{\sim}} = \theta_{a \vee a^{\sim}}$ (c) $\theta_a \cap \theta_b \subseteq \theta_{a \vee b}$. We will write $x\ \theta_a y$ to indicate $(x, y) \in \theta_a$

Proof:

(a) Since $a \wedge x = a \wedge x$, the relation is reflexive.

If $x\ \theta_a y$ then $a \wedge x = a \wedge y \Rightarrow a \wedge y = a \wedge x \Rightarrow y \theta_a x$ .Therefore the relation is symmetric

If $x\ \theta_a y$ and $y\ \theta_a z$ then $a \wedge x = a \wedge y$ and $a \wedge y = a \wedge z$.

This shows that $a \wedge x = a \wedge z \Rightarrow x \theta_a z$.

Therefore the relation is transitive. Hence the relation is equivalence relation.

If x, y, s, t $\in A$ satisfy $x\ \theta_a y$, $s\ \theta_a t$ then $a \wedge x = a \wedge y$, $a \wedge s = a \wedge t$

Now $a \wedge (x \vee s) = (a \wedge x) \vee (a \wedge s) = (a \wedge y) \vee (a \wedge t) = a \wedge (y \vee t)$.

This shows that $(x \vee s)\ \theta_a (y \vee t)$. Therefore $x\ \theta_a y, s\ \theta_a t$ then $(x \vee s)\ \theta_a (y \vee t)$.

Hence θ_a is closed under $\vee$. If x, y, s, t $\in A$ satisfy $x\ \theta_a y$, $s\ \theta_a t$ then

$a \wedge x = a \wedge y$, $a \wedge s = a \wedge t$. Consider $a \wedge (x \wedge s) = (a \wedge x) \wedge s = (a \wedge y) \wedge s = a \wedge (y \vee s)$

$= a \wedge (s \vee y) = (a \wedge s) \wedge y = (a \wedge t) \wedge y = a \wedge (t \wedge y) = a \wedge (y \wedge t)$.

This shows that $(x \wedge s)\ \theta_a (y \wedge t)$. Therefore $x\ \theta_a y, s\ \theta_a t$ then $(x \wedge s)\ \theta_a (y \wedge t)$.

Hence θ_a is closed under $\wedge$.

If x, y $\in A$ satisfy $x\ \theta_a y$, then
$a \wedge x = a \wedge y \Rightarrow a^{\sim} \vee x^{\sim} = a^{\sim} \vee y^{\sim} \Rightarrow a \wedge (a^{\sim} \vee x^{\sim}) = a \wedge (a^{\sim} \vee y^{\sim}) \Rightarrow a \wedge x^{\sim} = a \wedge y^{\sim} \Rightarrow x^{\sim}\ \theta_a y^{\sim}$.

Therefore θ_a is closed under $(-)^{\sim}$. Therefore θ_a is a congruence relation.

(b) If (x, y) $\in \theta_a \cap \theta_{a^{\sim}}$ then $a \wedge x = a \wedge y$ and $a^{\sim} \wedge x = a^{\sim} \wedge y$

Now $(a \vee a^{\sim}) \wedge x = (a \wedge x) \vee (a^{\sim} \wedge x) = (a \wedge y) \vee (a^{\sim} \wedge y) = (a \vee a^{\sim}) \wedge y$.

Therefore (x, y) $\in \theta_{a \vee a^{\sim}}$. This shows that $\theta_a \cap \theta_{a^{\sim}} \subseteq \theta_{a \vee a^{\sim}}$.

Let (x, y) $\in \theta_{a \vee a^{\sim}}$ then $(a \vee a^{\sim}) \wedge x = (a \vee a^{\sim}) \wedge y$.

Now $a \wedge x = \{a \wedge (a \vee a^{\sim})\} \wedge x = a \wedge (a \vee a^{\sim}) \wedge x = a \wedge (a \vee a^{\sim}) \wedge y = a \wedge y$.

This shows that (x, y) $\in \theta_a$. Again $a^{\sim} \wedge x = \{a^{\sim} \wedge (a^{\sim} \vee a)\} \wedge x = a^{\sim} \wedge (a^{\sim} \vee a) \wedge x$

$= a^{\sim} \wedge (a^{\sim} \vee a) \wedge y = a^{\sim} \wedge y$. This shows that (x, y) $\in \theta_{a^{\sim}}$. Therefore (x, y) $\in \theta_a \cap \theta_{a^{\sim}}$

Hence $\theta_a \cap \theta_{a^{\sim}} \supseteq \theta_{a \vee a^{\sim}}$. Hence as required.

(c) If (x, y) $\in \theta_a \cap \theta_b$ then $a \wedge x = a \wedge y$ and $b \wedge x = b \wedge y$

Now $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = (a \wedge y) \vee (b \wedge y) = (a \vee b) \wedge y$

This implies that $(x, y) \in \theta_{a \vee b}$. This shows that $\theta_a \cap \theta_b \subseteq \theta_{a \vee b}$.

2.3 Lemma :

Let A be a Pre A*-algebra and a, b $\in$B(A), then $\theta_a \cap \theta_b = \theta_{a \vee b}$

Proof :

By the lemma 2.2 we have $\theta_a \cap \theta_b \subseteq \theta_{a \vee b}$

Let us assume that $(x, y) \in \theta_{a\vee b}$, then $(a \vee b) \wedge x = (a \vee b) \wedge y$.

Now $a \wedge x = (a \wedge (a \vee b)) \wedge x$ (By lemma 1.10) $= a \wedge ((a \vee b) \wedge x)$

$= a \wedge ((a \vee b) \wedge y)$

$= (a \wedge (a \vee b)) \wedge y$

$= a \wedge y$

This implies that $(x, y) \in \theta_a$, similarly we can prove that $(x, y) \in \theta_b$

Therefore (x, y) $\in \theta_a \cap \theta_b$. This implies that $\theta_a \cap \theta_b \supseteq \theta_{a\vee b}$ .Hence as required.

2.4 Lemma :

Let A be a Pre A*-algebra, we let Δ_A denote the trivial congruence on A:

$\Delta_A = \{(x,x) / x \in A\}$ then

(a) $\theta_a = \Delta_A$ if and only if $a \wedge x = x$, $\forall$ x $\in$ A .

(b) $\theta_a = A \times A$ if and only if $a \wedge x = a$, $\forall$ x $\in$ A

(c) $(a,b) \in \theta_a, \theta_b$ then a=b.(d) If $a \in B(A)$ then $\theta_a \cap \theta_{a^\sim} = \Delta_A$

Proof:

(a) $(x, y) \in \theta_a \Leftrightarrow a \wedge x = a \wedge y \Leftrightarrow x = y \Leftrightarrow (x, y) \in \Delta_A$

(b) $a \wedge x = a$, $\forall$ x $\in$ A $\Leftrightarrow a \wedge x = a \wedge y$, $\forall x, y \in A$. This shows that $\theta_a = A \times A$

(c) $(a,b) \in \theta_a, \theta_b$ then $a \wedge a = b \wedge a$ & $a \wedge b = b \wedge b \Rightarrow a \wedge b = a$ & $a \wedge b = b \Rightarrow a = b$.

(d) If $a \in B(A)$ then $a \vee a^\sim = 1$. By lemma 2.2(b) $\theta_a \cap \theta_{a^\sim} = \theta_{a \vee a^\sim} \Rightarrow \theta_a \cap \theta_{a^\sim} = \theta_1 = \Delta_A$

2.5.Theorem.

Let A be a Pre A*-algebra, then A/θ = $\{\theta_a / a \in A\}$ is a Pre A*-algebra,

whose operations are defined as (i) $\theta_a \wedge \theta_b = \theta_{a\wedge b}$ (ii) $\theta_a \vee \theta_b = \theta_{a\vee b}$ (iii) $(\theta_a)^\sim = \theta_{a^\sim}$

Proof:

(a) $(\theta_a)^{\sim\sim} = (\theta_{a^\sim})^\sim = \theta_{a^{\sim\sim}} = \theta_a$

(b) For a , b $\in$A, $\theta_a \wedge \theta_a = \theta_{a\wedge a} = \theta_a$

(c) It is clear that $\theta_a \wedge \theta_b = \theta_b \wedge \theta_a$

(d) $(\theta_a \wedge \theta_b)^\sim = (\theta_{a\wedge b})^\sim = \theta_{(a\wedge b)^\sim} = \theta_{a^\sim \vee b^\sim} = \theta_{a^\sim} \vee \theta_{b^\sim} = \theta_a^\sim \vee \theta_b^\sim$

(e) One can easily verify that associate law with respect to $\wedge$

(f) It is clear that $\theta_a \wedge (\theta_b \vee \theta_c) = (\theta_a \wedge \theta_b) \vee (\theta_a \wedge \theta_c)$, for all a,b,c∈A

(g) $\theta_a \wedge (\theta_{a^\sim} \vee \theta_b) = \theta_a \wedge (\theta_{a^\sim \vee b}) = \theta_{a \wedge (a^\sim \vee b)} = \theta_{a \wedge b}$

Hence A/θ is a Pre A*-algebra.

3. TERNARY OPERATION

In this section we define the ternary operation $\Gamma_x(p,q) = (x \wedge p) \vee (x^\sim \wedge q)$ on Pre A*-algebra ($\Gamma_x(p,q)$ should be viewed as conditional " if x, then p, else q") , and derive the most important properties of the operation Γ. Also define $\psi_x = \{(p,q) \in A \times A / \Gamma_x(p,q) = p\}$ and establish the necessary and sufficient conditions for the above relation to be an equivalence relation.

3.1. Definition:

Let A be a Pre -A* algebra. If x, p, q $\in A$, define the ternary operation on A as

$$\Gamma_x(p,q) = (x \wedge p) \vee (x^\sim \wedge q)$$

3.2. Lemma :

Every Pre-A* algebra with the indicated constants satisfies the following laws.

(i) $\Gamma_2(p,q) = 2$ (ii) $\Gamma_x(2,2) = 2$ (iii) $\Gamma_1(p,q) = p \ (q \neq 2)$ (iv) $\Gamma_0(p,q) = q \ (p \neq 2)$

(v) $\Gamma_x(1,\ 0) = x$

Proof:

By inspection.

3.3. Lemma :

Every Pre-A* algebra satisfies the laws:

(i) $\Gamma_{x^\sim}(p,q) = \Gamma_x(q,p)$ (ii) $\Gamma_x(p,q) \wedge r = \Gamma_x(p \wedge r,\ q \wedge r)$

(iii) $\Gamma_x(p,q) \vee r = \Gamma_x(p \vee r,\ q \vee r)$ (iv) $\Gamma_x(\Gamma_y(p,q), \Gamma_y(r,s)) = \Gamma_y(\Gamma_x(p,r), \Gamma_x(q,s))$

(v) $\Gamma_{x \vee y}(p,q) = \Gamma_x(p, \Gamma_y(p,q))$ (vi) $\Gamma_{x \wedge y}(p,q) = \Gamma_x(\Gamma_y((p,q),q))$ (vii) $\Gamma_p(p,p) = p$

Proof:

(i) $\Gamma_{x^\sim}(p,q) = (x^\sim \wedge p) \vee (x^{\sim\sim} \wedge q) = (x^\sim \wedge p) \vee (x \wedge q)$

$= (x \wedge q) \vee (x^\sim \wedge p) = \Gamma_x(q,p)$

(ii) can be verified by applying distributive law.

(iii) If x=2, the result is true. If $x \neq 2$, then

$\Gamma_x(p\vee r,\ q\vee r)=\{x\wedge(p\vee r)\}\vee\{x^{\sim}\wedge(q\vee r)\}=\{(x\wedge p)\vee(x\wedge r)\}\vee\{(x^{\sim}\wedge q)\vee(x^{\sim}\wedge r)\}$

$=\{(x\wedge p)\vee(x^{\sim}\wedge q)\}\vee\{(x\wedge r)\vee(x^{\sim}\wedge r)\}$

$=\{(x\wedge p)\vee(x^{\sim}\wedge q)\}\vee\{(x\vee x^{\sim})\wedge r\}=\Gamma_x(p,q)\vee(1\wedge r)=\Gamma_x(p,q)\vee r$

(iv) $\Gamma_x(\Gamma_y(p,q),\Gamma_y(r,s))=\{x\wedge\Gamma_y(p,q)\}\vee\{x^{\sim}\wedge\Gamma_y(r,s)\}$

$=\{x\wedge[(y\wedge p)\vee(y^{\sim}\wedge q)]\}\vee\{x^{\sim}\wedge[(y\wedge r)\vee(y^{\sim}\wedge s)]\}$

$=\{x\wedge(y\wedge p)\}\vee\{x\wedge(y^{\sim}\wedge q)\}\vee\{x^{\sim}\wedge(y\wedge r)\}\vee\{x^{\sim}\wedge(y^{\sim}\wedge s)\}$

$=\{y\wedge(x\wedge p)\}\vee\{y\wedge(x^{\sim}\wedge r)\}\vee\{y^{\sim}\wedge(x\wedge q)\}\vee\{y^{\sim}\wedge(x^{\sim}\wedge s)\}$

$=[y\wedge\{(x\wedge p)\wedge(x^{\sim}\wedge r)\}]\vee[y^{\sim}\wedge\{(x\wedge q)\wedge\{x^{\sim}\wedge s)\}]$

$=[y\wedge\Gamma_x\{p,r)]\vee[y^{\sim}\wedge\Gamma_x(q,s)]=\Gamma_y(\Gamma_x(p,r),\Gamma_x(q,s))$

(v) $\Gamma_x(p,\Gamma_y(p,q))=(x\wedge p)\vee\{x^{\sim}\wedge\Gamma_y(p,q))=(x\wedge p)\vee[x^{\sim}\wedge\{(y\wedge p)\vee(y^{\sim}\wedge q)\}]$

$=(x\wedge p)\vee[\{x^{\sim}\wedge(y\wedge p)\}\vee\{x^{\sim}\wedge(y^{\sim}\wedge q)\}]=[(x\wedge p)\vee\{x^{\sim}\wedge(y\wedge p)\}]\vee\{x^{\sim}\wedge(y^{\sim}\wedge q)\}]$

$=\{(x\vee y)\wedge p\}\vee\{(x\vee y)^{\sim}\wedge q\}=\Gamma_{x\vee y}(p,q)$

(vi) $\Gamma_x(\Gamma_y((p,q),q))=(x\wedge\Gamma_y(p,q))\vee(x^{\sim}\wedge q)=[x\wedge\{(y\wedge p)\vee(y^{\sim}\wedge q)\}\vee(x^{\sim}\wedge q)]$

$=[(x\wedge(y\wedge p))\vee(x\wedge(y^{\sim}\wedge q))]\vee(x^{\sim}\wedge q)=(x\wedge(y\wedge p))\vee[(x\wedge(y^{\sim}\wedge q))\vee(x^{\sim}\wedge q)]$

$=\{(x\wedge y)\wedge p\}\vee\{(x^{\sim}\vee y^{\sim})\wedge q\}=\{(x\wedge y)\wedge p\}\vee\{(x\wedge y)^{\sim}\wedge q\}=\Gamma_{x\wedge y}(p,q)$

(vii) $\Gamma_p(p,p)=(p\wedge p)\vee(p^{\sim}\wedge p)=p\vee\{p^{\sim}\wedge p)=p$

3.4 Definition :

Let A be a Pre A*- algebra and $x\in A$ .Define

$\psi_x=\{(p,q)\in A\times A/\Gamma_x(p,q)=p\}$

3.5 Lemma :

Let A be a Pre A*-algebra with $1\in A$, $x\in B(A)$ and $p,q,r\in A$, then

$\Gamma_x(p,q)=r$ if and only if $x\wedge r=x\wedge p$, $x^{\sim}\wedge r=x^{\sim}\wedge q$

Proof:

Suppose that $\Gamma_x(p,q)=r$.

Now $x\wedge r=x\wedge\Gamma_x(p,q)=x\wedge\{(x\wedge p)\vee(x^{\sim}\wedge q)\}=(x\wedge x\wedge p)\vee(x\wedge x^{\sim}\wedge q)$

$=(x\wedge p)\vee(0\wedge q)=x\wedge p$ (provided $q\neq 2$).

If q =2 then r =2 and also $x\wedge r=x\wedge p$, and $x^{\sim}\wedge r=x^{\sim}\wedge\Gamma_x(p,q)$

$= x^{\sim} \wedge \{(x \wedge p) \vee (x^{\sim} \wedge q)\} = (x^{\sim} \wedge x \wedge p) \vee (x^{\sim} \wedge x^{\sim} \wedge q) = (0 \wedge p) \vee (x^{\sim} \wedge q)$

$= x^{\sim} \wedge q$ (provided $p \neq 2$).

If p = 2 then r = 2 and also $x^{\sim} \wedge r = x^{\sim} \wedge q$

Conversely suppose that $x \wedge r = x \wedge p$, $x^{\sim} \wedge r = x^{\sim} \wedge q$

$\Gamma_x(p,q) = (x \wedge p) \vee (x^{\sim} \wedge q) = (x \wedge r) \vee (x^{\sim} \wedge r) = (x \vee x^{\sim}) \wedge r = 1 \wedge r = r$

3.6. Lemma :

Let A be a Pre A*-algebra with $1 \in A$, $x \in B(A)$ and $p,q,r \in A$, then

$\Gamma_x(q,p) = r$ then $x \vee r = x \vee p$, $x^{\sim} \vee r = x^{\sim} \vee q$

Proof :

Suppose $\Gamma_x(q,p) = r$.

Now $x \vee r = x \vee \{(x \wedge q) \vee (x^{\sim} \wedge p)\} = \{x \vee (x \wedge q)\} \vee (x^{\sim} \wedge p)$

$= x \vee (x^{\sim} \wedge p)$ (provided $q \neq 2$) $= x \vee p$

If q=2 then r =2 and also $x \vee r = x \vee q$.

and $x^{\sim} \vee r = x^{\sim} \vee \{(x \wedge q) \vee (x^{\sim} \wedge p)\} = \{x^{\sim} \vee (x^{\sim} \wedge p)\} \vee (x \wedge q)$ (provided $p \neq 2$)

$= x^{\sim} \vee (x \wedge q) = x^{\sim} \vee q$

If p =2 then r =2 and also $x^{\sim} \vee r = x^{\sim} \vee p$

3.7. Theorem :

Let A be a Pre A* algebra with 1 and $x \in B(A)$ then following are equivalent :(a) $x \in B(A)$, (b) $\psi_x = \theta_{x^{\sim}}$, (c) $\theta_{x^{\sim}} \subseteq \psi_x$, $\theta_{x^{\sim}} \supseteq \psi_x$,

(d) ψ_x is congruence relation on A, (e) ψ_x is reflexive on A.(f) ψ_x is symmetric on A.

Proof :

$a \Rightarrow b$, suppose $x \in B(A)$, then by lemma 3.5, we have $\Gamma_x(p,q) = p$ if and only if $x^{\sim} \wedge p = x^{\sim} \wedge q$, for all $p,q \in A$. This shows that $\psi_x = \theta_{x^{\sim}}$

$b \Rightarrow c$ is clear

$c \Rightarrow d$ follows by lemma 2.2(a)

$d \Rightarrow e$ is clear

$e \Rightarrow f$ suppose that ψ_x is reflexive on B(A), $p,q \in A$ and $(p,q) \in \psi_x$ then $(p,q) \in$, $Q_{x^{\sim}}$ so that $x^{\sim} \wedge p = x^{\sim} \wedge q$

Now $\Gamma_x(q,p)=(x\wedge q)\vee(x^{\sim}\wedge p)=(x\wedge q)\vee(x^{\sim}\wedge q)$

$=\Gamma_x(q,q)=q$ (since ψ_x is reflexive on A).

Therefore $(q,p)\in\psi_x$ and hence ψ_x is symmetric on A

$f\Rightarrow a$ suppose that ψ_x is symmetric on A.

$\Gamma_x(x\vee x^{\sim},1)=(x\wedge(x\vee x^{\sim}))\vee(x^{\sim}\wedge 1)=x\vee(x^{\sim}\wedge 1)=x\vee x^{\sim}$.

$\therefore$ $(x\vee x^{\sim},1)\in\psi_x$ and hence $(1,x\vee x^{\sim})\in\psi_x$

$\Rightarrow 1=\Gamma_x(1,x\vee x^{\sim})=(x\wedge 1)\vee(x^{\sim}\wedge(x\vee x^{\sim}))=x\vee x^{\sim}$. This implies that $x\in B(A)$.

3.8. Lemma :

Let A be Pre A*-algebra with 1 and a , b $\in B(A)$ then

(a) $\theta_a \circ \theta_b=\theta_{a\wedge b}$, (b) $\theta_a \circ \theta_b=\theta_b \circ \theta_a$,

(c) $\theta_a \subseteq \theta_b$ if and only if $b=a\wedge b$, (d) $\theta_a \circ \theta_{a^{\sim}}=A\times A$

Proof:

(a) By lemma 2.3 $\theta_a\cap\theta_{a\wedge b}=\theta_{a\vee(a\wedge b)}=\theta_a$ (By lemma1.10(b))

Therefore $\theta_a\subseteq\theta_{a\wedge b}$. By symmetry $\theta_b\subseteq\theta_{a\wedge b}$, which imply that $\theta_a \circ \theta_b\subseteq\theta_{a\wedge b}$.

Suppose that $(p,q)\in\theta_{a\wedge b}$, then $(a\wedge b)\wedge p=(a\wedge b)\wedge q$

Put $r=\Gamma_b(p,q)$. By lemma 3.7, $b\wedge r=b\wedge p$. Therefore $(p,r)\in\theta_b$, and $a\wedge r=a\wedge\{(b\wedge p)\vee(b^{\sim}\wedge q)\}=(a\wedge b\wedge p)\vee(a\wedge b^{\sim}\wedge q)=(a\wedge b\wedge q)\vee(a\wedge b^{\sim}\wedge q)$

$=a\wedge\{(b\vee b^{\sim})\wedge q\}=a\wedge(1\wedge q)=a\wedge q$. Therefore $(r,q)\in\theta_a$.

Hence $(p,q)\in\theta_a \circ \theta_b$ which imply that $\theta_a \circ \theta_b\supseteq\theta_{a\wedge b}$. Hence as required.

(b) By (a) we have $\theta_a \circ \theta_b=\theta_{a\wedge b}=\theta_{b\wedge a}=\theta_b \circ \theta_a$

(c) Suppose that $\theta_a \subseteq \theta_b$, then $(1,a)\in\theta_a\subseteq\theta_b\Rightarrow(1,a)\in\theta_b$

$\Rightarrow 1\wedge b=a\wedge b \Rightarrow b=a\wedge b$.

Conversely suppose that $b=a\wedge b$, from (a) $\theta_a \circ \theta_b=\theta_{a\wedge b}=\theta_b$ which imply that

$\theta_a \subseteq \theta_b$.

(d) $\theta_a \circ \theta_{a^{\sim}}=\theta_{a\wedge a^{\sim}}=\theta_0=A\times A$

3.9. Definition :

A congruence θ on an algebra A is called a factor congruence on A if there is a congruence ϕ on A such that $\theta\cap\phi=\Delta_A$ and $\theta\circ\phi=A\times A$.

3.10. Theorem:

Let A be a Pre A*-algebra with 1 and θ be congruence on A. If $\theta = \theta_x$ for some $x \in B(A)$, then θ_x is a factor congruence on A

Proof:

Suppose $\theta = \theta_x, for$ some $x \in B(A)$. Then by lemma 2.4 (d) and lemma 3.9(d) we have $\theta_x \cap \theta_{x^-} = \Delta_A$ and θ_x o θ_{x^-} =$A \times A$. Thus θ_x is a factor congruence on A.

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