On Some Double Generating Functions of Jacobi polynomials of Three Variables

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ABSTRACT

The present paper is a study of double generating functions of Jacobi poly- nomials of three variables.

Key words: Double generating functions.

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1. INTRODUCTION

In our earlier paper [2] we obtained some generating functions of Jacobi polynomials of three variables. In the present paper we have obtained some double generating functions of Jacobi polynomials of three variables. In our investigation we require the following definitions: Jacobi polynomials of three variables are defined by [5]

$$p_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2;\,\alpha_3,\,\beta_3)}(x,y,z) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}{(n!)^3}$$

$$\times F_A^{(3)}\left[-n, 1+\alpha_1+\beta_1+n, 1+\alpha_2+\beta_2+n, 1+\alpha_3+\beta_3+n; 1+\alpha_1, 1+\alpha_2, 1+\alpha_3; \frac{1-x}{2}, \frac{1-y}{2}, \frac{1-z}{2}\right], \qquad (1.1)$$

where $F_A^{(3)}$ denotes one of the Lauricella's hypergeometric functions of three variables [4,p. 60] as follows:

$$F_A^{(3)}[a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z] = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a)_{m_1+m_2+m_3}(b_1)_{m_1}(b_2)_{m_2}(b_3)_{m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\frac{x_3^{m_3}}{m_3!}, \qquad (1.2)$$

$$|x_1| + |x_2| + |x_3| < 1.$$

The confluent hypergeometric functions of two variables are dened by [4]

$$\phi_1[\alpha, \beta; \gamma; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad |x| < 1, \quad |y| < \infty; \tag{1.3}$$

$$\phi_2[\beta, \beta'; \gamma; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad |x| < \infty, \quad |y| < \infty; \tag{1.4}$$

$$\phi_3[\beta; \gamma; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \qquad |x| < \infty, \quad |y| < \infty; \tag{1.5}$$

$$\Psi_1[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad |x| < 1, \quad |y| < \infty; \tag{1.6}$$

$$\Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad |x| < \infty, \quad |y| < \infty; \tag{1.7}$$

$$\Xi_1[\alpha, \alpha', \beta; \gamma; x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad |x| < 1, \quad |y| < \infty. \tag{1.8}$$

2. DOUBLE GENERATING FUNCTIONS

The polynomials $P_n^{(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)}(x, y, z)$ admits the following double generating functions:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n} P_n^{(\alpha_1, \beta_1+m-n; \alpha_2, \beta_2-n; \alpha_3, \beta_3-n)}(x, y, z) t^n u^m$$

$$= e^t (1-u)^{-1-\alpha_1-\beta_1} {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right] {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right], \tag{2.1}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\,(1-u)^{-1-\alpha_2-\beta_2}\,{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]$$

$$\times{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{\frac{1}{2}(y-1)t}{(1-u)}\right]\,{}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right], \qquad (2.2)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$=e^t\,(1-u)^{-1-\alpha_3-\beta_3}\,{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]$$

$$\times{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\,{}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{\frac{1}{2}(z-1)t}{(1-u)}\right], \qquad (2.3)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1+m)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\,{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\,{}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times\phi_1\left[1+\alpha_1+\beta_1,1+\alpha_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \qquad (2.4)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2+m)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\,{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\,{}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times\phi_1\left[1+\alpha_2+\beta_2,1+\alpha_2;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \qquad (2.5)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3+m)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\phi_1\left[1+\alpha_3+\beta_3,1+\alpha_3;1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \tag{2.6}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.7}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.8}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\phi_1\left[1+\alpha_3+\beta_3,\lambda;1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \tag{2.9}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1+m)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_2\left[1+\alpha_1,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.10}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2+m)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_2\left[1+\alpha_2,1+\alpha_2+\beta_2;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.11}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3+m)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-m-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\phi_2\left[1+\alpha_3,1+\alpha_3+\beta_3;1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \tag{2.12}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_2\left[\lambda,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.13}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_2\left[\lambda,1+\alpha_2+\beta_2;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.14}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-m-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\phi_2\left[\lambda,1+\alpha_3+\beta_3;1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \tag{2.15}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right], \tag{2.16}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,u\right]. \tag{2.17}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-m-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\phi_3\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t,u\right], \tag{2.18}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.19}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2+\beta_2,1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.20}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$=e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_3+\beta_3,\lambda;1+\alpha_3+\beta_3,1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \tag{2.21}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_1+\beta_1,\lambda;\mu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.22}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_2+\beta_2)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2,\lambda;\mu,1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.23}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_3+\beta_3)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_3+\beta_3,\lambda;\mu,1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \qquad (2.24)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_1+\beta_1; 1+\alpha_1+\beta_1, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \qquad (2.25)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_2+\beta_2; 1+\alpha_2+\beta_2, 1+\alpha_2; u, \frac{1}{2}(y-1)t\right], \qquad (2.26)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_3+\beta_3; 1+\alpha_3+\beta_3, 1+\alpha_3; u, \frac{1}{2}(z-1)t\right], \qquad (2.27)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(\lambda)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_1+\beta_1; \lambda, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \qquad (2.28)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m}{m!(\lambda)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\Psi_2\left[1+\alpha_2+\beta_2;\lambda,1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.29}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_3+\beta_3)_m}{m!(\lambda)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\Psi_2\left[1+\alpha_3+\beta_3;\lambda,1+\alpha_3;u,\frac{1}{2}(z-1)t\right], \tag{2.30}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.31}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$
$$\times\Xi_1\left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;u,\frac{1}{2}(y-1)t\right], \tag{2.32}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(\mu)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-m-n)}(x,y,z)t^n u^m$$
$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$
$$\times\Xi_1\left[\lambda,1+\alpha_3+\beta_3,\mu;1+\alpha_3;u,\frac{1}{2}(z-1)t\right]. \tag{2.33}$$

PROOF OF (2.1) :

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}\times\frac{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}{(n!)^3}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s}\frac{(-n)_{r+s+k}(1+\alpha_1+\beta_1+m)_r(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{r!s!k!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s\left(\frac{1-z}{2}\right)^k t^n u^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s}\frac{(-1)^{r+s+k}n!(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{m!n!r!s!k!(n-r-s-k)!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s\left(\frac{1-z}{2}\right)^k t^n u^m$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{m!n!r!s!k!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times\left(\frac{x-1}{2}\right)^r\left(\frac{y-1}{2}\right)^s\left(\frac{z-1}{2}\right)^k t^{n+r+s+k}u^m$$

$$=e^t\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{r!s!k!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k}\left(\frac{1}{2}(x-1)t\right)^r$$

$$\times\left(\frac{1}{2}(y-1)t\right)^s\left(\frac{1}{2}(z-1)t\right)^k\sum_{m=0}^{\infty}\frac{(1+\alpha_1+\beta_1+r)_m}{m!}u^m$$

$$= e^t(1-u)^{-1-\alpha_1-\beta_1} \sum_{r=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_r}{r!(1+\alpha_1)_r} \left(\frac{\frac{1}{2}(x-1)t}{(1-u)}\right)^r \sum_{s=0}^{\infty} \frac{(1+\alpha_2+\beta_2)_s}{s!(1+\alpha_2)_s} \left(\frac{1}{2}(y-1)t\right)^s$$

$$\times \sum_{k=0}^{\infty} \frac{(1+\alpha_3+\beta_3)_k}{k!(1+\alpha_3)_k} \left(\frac{1}{2}(z-1)t\right)^k$$

$$= e^t(1-u)^{-1-\alpha_1-\beta_1} {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right] {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]$$

$$\times {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

which proves (2.1).

The proof of results (2.2) and (2.3) are similar to that of (2.1).

PROOF OF (2.4):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1+m)_n(1+\alpha_2)_n(1+\alpha_3)_n} P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1+m)_n(1+\alpha_2)_n(1+\alpha_3)_n} \times \frac{(1+\alpha_1+m)_n(1+\alpha_2)_n(1+\alpha_3)_n}{(n!)^3}$$

$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}(1+\alpha_1+\beta_1+m)_r(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{r!s!k!(1+\alpha_1+m)_r(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times \left(\frac{1-x}{2}\right)^r \left(\frac{1-y}{2}\right)^s \left(\frac{1-z}{2}\right)^k t^n u^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-1)^{r+s+k} n!(1+\alpha_1)_m(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{m!n!r!s!k!(n-r-s-k)!(1+\alpha_1)_{m+r}(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times \left(\frac{1-x}{2}\right)^r \left(\frac{1-y}{2}\right)^s \left(\frac{1-z}{2}\right)^k t^n u^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(1+\alpha_1)_m(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_s(1+\alpha_3+\beta_3)_k}{m!n!r!s!k!(1+\alpha_1)_{m+r}(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times \left(\frac{x-1}{2}\right)^r \left(\frac{y-1}{2}\right)^s \left(\frac{z-1}{2}\right)^k t^{n+r+s+k} u^m$$

$$= e^t \sum_{s=0}^{\infty} \frac{(1+\alpha_2+\beta_2)_s}{s!(1+\alpha_2)_s} \left(\frac{1}{2}(y-1)t\right)^s \sum_{k=0}^{\infty} \frac{(1+\alpha_3+\beta_3)_k}{k!(1+\alpha_3)_k} \left(\frac{1}{2}(z-1)t\right)^k$$

$$\times \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_1)_m}{m!r!(1+\alpha_1)_{m+r}} u^m \left(\frac{1}{2}(x-1)t\right)^r$$

where ϕ_1 is (Humbert's) confluent hypergeometric function of two variables defined by (1.3), which proves (2.4).

The proof of results (2.5) to (2.33) are similar to that of (2.4).

Further,we have the following double generating functions:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n} P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : -: & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_1; 1+\alpha_2; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \tag{2.34}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n} P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_3+\beta_3 : -: & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_1; 1+\alpha_3; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(z-1)t\right], \tag{2.35}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_2+\beta_2; -; 1+\alpha_3+\beta_3 : -; & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_2; 1+\alpha_3; \end{matrix}\ u, \frac{1}{2}(y-1)t, \frac{1}{2}(z-1)t\right], \tag{2.36}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : \lambda; & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : \mu; 1+\alpha_1; 1+\alpha_2; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \tag{2.37}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_3+\beta_3)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_3+\beta_3 : \lambda; & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : \mu; 1+\alpha_1; 1+\alpha_3; \end{matrix}\ u, \frac{1}{2}(x-1)t, \frac{1}{2}(z-1)t\right], \tag{2.38}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_m(1+\alpha_2+\beta_2)_m(1+\alpha_3+\beta_3)_m}{m!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2,\,\beta_2+m-n;\,\alpha_3,\,\beta_3+m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_2+\beta_2; -; 1+\alpha_3+\beta_3 : \lambda; & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : \mu; 1+\alpha_2; 1+\alpha_3; \end{matrix}\ u, \frac{1}{2}(y-1)t, \frac{1}{2}(z-1)t\right], \tag{2.39}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^nu^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} -::1+\alpha_1+\beta_1;-;1+\alpha_2+\beta_2:-;-;-; \\ -::\quad 1+\alpha_1\quad :-:\quad 1+\alpha_2\quad :-;-;-;\end{matrix}\ u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \qquad (2.40)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-n)}(x,y,z)t^nu^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} -::1+\alpha_1+\beta_1;-;1+\alpha_3+\beta_3:-:-;-; \\ -::\quad 1+\alpha_1\quad ;-;\quad 1+\alpha_3\quad :-:-;-;\end{matrix}\ u,\frac{1}{2}(x-1)t,\frac{1}{2}(z-1)t\right], \qquad (2.41)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-n)}(x,y,z)t^nu^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} -::1+\alpha_2+\beta_2;-:1+\alpha_3+\beta_3:-:-;-; \\ -::\quad 1+\alpha_2\quad :-:\quad 1+\alpha_3\quad :-:-;-;\end{matrix}\ u,\frac{1}{2}(y-1)t,\frac{1}{2}(z-1)t\right], \qquad (2.42)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^nu^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_3+\beta_3;1+\alpha_3;\frac{1}{2}(z-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} -::1+\alpha_1+\beta_1;-;\quad -\quad :-;-;1+\alpha_2+\beta_2; \\ -::\quad 1+\alpha_1\quad :-:1+\alpha_2:-;-:\quad -\quad :\end{matrix}\ u,\frac{1}{2}(x-1)t,\frac{1}{2}(y-1)t\right], \qquad (2.43)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_1+\beta_1)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-m-n)}(x,y,z)t^nu^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; & - & : -; -; 1+\alpha_3+\beta_3; \\ - :: & 1+\alpha_1 & : -; 1+\alpha_3 : -; -; & - & ; \end{matrix} u, \frac{1}{2}(x-1)t, \frac{1}{2}(z-1)t\right]. \tag{2.44}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+m,\,\beta_2-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: & - & ; -; 1+\alpha_2+\beta_2 : -; 1+\alpha_1+\beta_1; -; \\ - :: 1+\alpha_1; -: & & 1+\alpha_2 : -; & - & ; -; \end{matrix} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right]. \tag{2.45}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_2+\beta_2; -; & - & : -; -; 1+\alpha_3+\beta_3; \\ - :: & 1+\alpha_2 & ; -; 1+\alpha_3 : -; -; & - & ; \end{matrix} u, \frac{1}{2}(y-1)t, \frac{1}{2}(z-1)t\right]. \tag{2.46}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_{m+n}}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2-n;\,\alpha_3+m,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: & - & ; -; 1+\alpha_3+\beta_3 : -; 1+\alpha_1+\beta_1; -; \\ - :: 1+\alpha_1; -; & & 1+\alpha_3 : -; & - & ; -; \end{matrix} u, \frac{1}{2}(x-1)t, \frac{1}{2}(z-1)t\right]. \tag{2.47}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2(1+\alpha_3+\beta_3)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_{m+n}}P_n^{(\alpha_1,\,\beta_1-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3+m,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t\right]$$

$$\times F^{(3)}\left[\begin{matrix} - :: & - & ; -; 1+\alpha_3+\beta_3 : -; 1+\alpha_2+\beta_2; -; \\ - :: 1+\alpha_2; -; & & 1+\alpha_3 : -; & - & ; -; \end{matrix} u, \frac{1}{2}(y-1)t, \frac{1}{2}(z-1)t\right]. \tag{2.48}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n!)^2}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_{m+n}(1+\alpha_3)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+m,\,\beta_2-m-n;\,\alpha_3,\,\beta_3-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right]$$

$$\times F^{(3)}\left[\begin{array}{l} -:: \quad - ; -; \quad - : -; 1+\alpha_1+\beta_1; 1+\alpha_2+\beta_2; \\ -:: 1+\alpha_1; -; 1+\alpha_2 : -; \quad - \quad ; \quad - \quad ; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right], \tag{2.49}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2}{m!(1+\alpha_1)_{m+n}(1+\alpha_2)_n(1+\alpha_3)_{m+n}} P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2,\ \beta_2-n;\ \alpha_3+m,\ \beta_3-m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t\right]$$

$$\times F^{(3)}\left[\begin{array}{l} -:: \quad - ; -; \quad - : -; 1+\alpha_1+\beta_1; 1+\alpha_3+\beta_3; \\ -:: 1+\alpha_1; -; 1+\alpha_3 : -; \quad - \quad ; \quad - \quad ; \end{array} u, \frac{1}{2}(x-1)t, \frac{1}{2}(z-1)t\right], \tag{2.50}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2}{m!(1+\alpha_1)_n(1+\alpha_2)_{m+n}(1+\alpha_3)_{m+n}} P_n^{(\alpha_1,\ \beta_1-n;\ \alpha_2+m,\ \beta_2-m-n;\ \alpha_3+m,\ \beta_3-m-n)}(x,y,z)t^n u^m$$

$$= e^t\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t\right]$$

$$\times F^{(3)}\left[\begin{array}{l} -:: \quad - ; -; \quad - : -; 1+\alpha_2+\beta_2; 1+\alpha_3+\beta_3; \\ -:: 1+\alpha_2; -; 1+\alpha_3 : -; \quad - \quad ; \quad - \quad ; \end{array} u, \frac{1}{2}(y-1)t, \frac{1}{2}(z-1)t\right], \tag{2.51}$$

Proof of (2.34) :

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n} P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2,\ \beta_2+m-n;\ \alpha_3,\ \beta_3-n)}(x,y,z)t^n u^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n!)^2(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_m}{m!(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n} \times \frac{(1+\alpha_1)_n(1+\alpha_2)_n(1+\alpha_3)_n}{(n!)^3}$$

$$\times \sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}(1+\alpha_1+\beta_1+m)_r(1+\alpha_2+\beta_2+m)_s(1+\alpha_3+\beta_3)_k}{r!s!k!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k}$$

$$\times \left(\frac{1-x}{2}\right)^r \left(\frac{1-y}{2}\right)^s \left(\frac{1-z}{2}\right)^k t^n u^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-1)^{r+s+k} n!(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{m+s}(1+\alpha_3+\beta_3)_k}{m!n!r!s!k!(n-r-s-k)!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k} \times \left(\frac{1-x}{2}\right)^r \left(\frac{1-y}{2}\right)^s \left(\frac{1-z}{2}\right)^k t^n u^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{m+s}(1+\alpha_3+\beta_3)_k}{m!n!r!s!k!(1+\alpha_1)_r(1+\alpha_2)_s(1+\alpha_3)_k} \left(\frac{x-1}{2}\right)^r \times \left(\frac{y-1}{2}\right)^s \left(\frac{z-1}{2}\right)^k t^{n+r+s+k} u^m$$

$$= e^t \sum_{k=0}^{\infty} \frac{(1+\alpha_3+\beta_3)_k}{k!(1+\alpha_3)_k} \left(\frac{1}{2}(z-1)t\right)^k \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{s+m}}{m!r!s!(1+\alpha_1)_r(1+\alpha_2)_s} u^m \times \left(\frac{1}{2}(x-1)t\right)^r \left(\frac{1}{2}(y-1)t\right)^s$$

$$= e^t \; {}_1F_1\left[1+\alpha_3+\beta_3; 1+\alpha_3; \frac{1}{2}(z-1)t\right] \times F^{(3)}\left[\begin{matrix} - :: 1+\alpha_1+\beta_1; -; 1+\alpha_2+\beta_2 : -; & - ; & - ; \\ - :: \quad - \quad ; -; \quad - \quad : -; 1+\alpha_1; 1+\alpha_2; \end{matrix}\; u, \frac{1}{2}(x-1)t, \frac{1}{2}(y-1)t\right]$$

where $F^3[x; y; z]$ denotes a general triple hypergeometric series [4, eq.39, p.69],which proves (2.34).

The proof of results (2.35) to (2.51) are similar to that of (2.34).

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