n-Fuzzy Proximity – III
Fuzzy Topology Induced By n-Fuzzy Proximity

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Abstract

In this paper, the fuzzy topology induced by n-fuzzy proximity is defined and a relation between the topologies induced by n-fuzzy proximity and its extension is obtained.

Key Words : Fuzzy topology, n-fuzzy proximity, extension of n-fuzzy proximity, I_n-valued fuzzy sets, n^{th} upper approximation and n^{th} lower approximation and n^{th} lower approximation of a fuzzy set topology induced by proximity.

Mathematics Subject Classification : **54E05, 54E15.**

1. Introduction

The concept of fuzzy topology was first introduced by Chang, C.L. [1]. Various notions in general topology were extended to fuzzy topology by many authors. In 1979, Katsaras [3] introduced the first definition of fuzzy proximity and the fuzzy topology induced by the fuzzy proximity. Jayalakshmi [2] introduced n^{th} order approximations of fuzzy sets. Sivakamasundari [4] introduced n-fuzzy proximity ρ_{n*} and the extension of it to a fuzzy proximity $E_x(\rho_{n*})$. The concept of fuzzy proximity base was introduced by Srivastava and Gupta [6] in 1980. Sivakamasundari [5] introduced n-fuzzy proximity base and product. In this paper n-fuzzy topology induced by n-fuzzy proximity is defined.

Here n-fuzzy proximity ρ_{n*} and its extension, the fuzzy proximity, $E_x(\rho_{n*})$ are used to prove that the topology $\delta(\rho_{n*})$ induced by n-fuzzy proximity ρ_{n*} is the same as the topology $\delta(E_x(\rho_{n*}))$ induced by the fuzzy proximity $E_x(\rho_{n*})$.

2. Preliminary Results

Definition 2.1

Let $I_n = \{0, 1/n, 2/n, \ldots, 1\}$. A **$I_n$-valued fuzzy set** on X is an element of the set I_n^X of all functions from X to I_n.

Definition 2.2 [4]

A binary relation ρ_{n*} on I_n^X is called an **n-fuzzy proximity on X** if ρ_{n*} satisfies the following axioms.

For any f, g, h $\in I_n^X$.

(F P_{n^*} 1) $f \rho_{n^*} g$ implies $g \rho_{n^*} f$

(F P_{n^*} 2) $(f \vee h) \rho_{n^*} g$ iff $f \rho_{n^*} g$ or $h \rho_{n^*} g$

(F P_{n^*} 3) $f \rho_{n^*} g$ implies $f \neq \mathbf{0}$ and $g \neq \mathbf{0}$

(F P_{n^*} 4) $f \bar{\rho}_{n^*} g$ implies that there exists an $A \subseteq X$ such that $f \bar{\rho}_{n^*} \chi_A$ and $(1 - \chi_A) \bar{\rho}_{n^*} g$

(F P_{n^*} 5) $f \wedge g \neq \mathbf{0}$ implies $f \rho_{n^*} g$

The pair (X, ρ_{n^*}) is called an **n-fuzzy proximity space.**

To extend the concept of n-fuzzy proximity to a fuzzy proximity on X, the concept of nth order approximation introduced in [2] is required. The definition and properties of nth order approximations are collected below.

Definition 2.3

With every fuzzy set f defined on a set X and with every positive integer n, a finite fuzzy set ${}^n f$ with values in I_n is associated as follows :

For $x \in X$

(i) if $f(x) = 0$, define ${}^n f(x) = 0$.

(ii) if $1/n < f(x) \leq (i+1)/n$ define ${}^n f(x) = (i+1)/n$, for $i = 0, 1, 2, \ldots, n-1$.

${}^n f$ is called the **nth upper approximation of f.**

Proposition 2.4

(i) $f(x) = i/n \Rightarrow {}^n f(x) = i/n$ for $i = 1, 2, \ldots, n$

(ii) For all n, $f \leq {}^n f$.

(iii) $f \leq g \Rightarrow {}^n f \leq {}^n g$

(iv) $f \leq {}^n g \Rightarrow {}^n f \leq {}^n g$

(v) ${}^n({}^n f) = {}^n f$

(vi) ${}^n(\vee f_\lambda) = \vee({}^n f_\lambda)$

(vii) ${}^n(\bigwedge_{k=1}^{m} f_k) = \bigwedge_{k=1}^{m}({}^n f_k)$

Definition 2.5

For each fuzzy set f on a set X, the **nth lower approximation** $_n f$ is defined as follows :

For $x \in X$,

(i) if $f(x) = 1$ define $_n f(x) = 1$

(ii) if $i/n \leq f(x) < (i+1)/n$, define $_n f(x) = i/n$ for $i = 0, 1, 2, \ldots, n-1$.

Proposition 2.6

(i) If $f(x) = i/n$ then $_n f(x) = i/n$, for $i = 0, 1, \ldots, n-1$

(ii) $_n f(x) \leq f(x)$ for all n.

(iii) $f \leq g \Rightarrow {}_n f \leq {}_n g$.

(iv) $_n f \leq g \Rightarrow {}_n f \leq {}_n g$

(v) $_n({}_n f) = {}_n f$

(vi) $_n(\wedge f_\lambda) = \wedge ({}^n f_\lambda)$

(vii) $_n(\overset{m}{\underset{k=1}{\vee}} f_k) = \overset{m}{\underset{k=1}{\vee}} ({}_n f_k)$

Proposition 2.7

(i) $_n(1 - f) = 1 - {}^n f$ and ${}^n(1 - f) = 1 - {}_n f$

(ii) ${}^n f \leq g \Rightarrow {}^n f \leq {}_n g$

(iii) $f \leq {}_n g \Rightarrow {}^n f \leq {}_n g$, $f \leq {}^n g \Rightarrow {}_n f \leq g \leq {}^n g$

(iv) $_n({}^n f) = {}^n f$

(v) ${}^n({}_n f) = {}_n f$

(vi) ${}^n f \neq \mathbf{0} \Rightarrow f \neq \mathbf{0}$

Proposition 2.8

(i) If $f \in I_n^X$ then $_n f = f = {}^n f$

(ii) For $A \subseteq X$, ${}^n\chi_A = {}_n\chi_A$

Proposition 2.9 [4]

Given an n-fuzzy proximity ρ_{n*}, it is extended to a fuzzy proximity $\mathbf{E}_x(\rho_{n*})$ as follows :

$f\,(E_x\,(\rho_{n*}))\,g \Leftrightarrow {}^n f\ \rho_{n*}\ {}^n g$. Here $E_x\,(\rho_{n*})$ is called **the extension of** ρ_{n*}.

3. Fuzzy Topology Induced By N-Fuzzy Proximity

Definition [Katsaras, 3] 3.1

Let (X, ρ) be a fuzzy proximity space. For $f \in I^X$ define $\mathbf{cl\, f = 1 - \vee \{g \in I^X \mid f\ \rho\ g\}}$.

The map $f \to \text{cl } f$ is a closure operator on I^X. The collection

$\delta(\rho) = \{f \in I^X \mid \text{cl}\,(1 - f) = (1 - f)\}$ is a fuzzy topology on X and it is called **the fuzzy topology induced by** ρ.

Definition 3.2

Let (X, ρ_n^*) be an n-fuzzy proximity space. For $f \in I_n^X$ define

$\text{Kl } f = 1 - \vee\{g \in I_n^X \mid f\, \bar{\rho}_{n^*}\, g\}$.

Proposition 3.3

The map $f \to \text{Kl } f$ is a closure operator on I_n^X.

Proof

(a) $\text{Kl}\,(0) = 0$, $\text{Kl}\,(1) = 1$

(b) To prove $f \leq \text{Kl}\,(f)$

Let $x \in X$, for $f, g \in I_n^X$ if $g\, \bar{\rho}_{n^*}\, f$, then $g \wedge f = 0$ and hence either $g(x) = 0$ or $f(x) = 0$.

In both the cases we have $g(x) \leq 1 - f(x)$. Thus $\sup\limits_{g \bar{\rho}_{n^*} f} g(x) \leq 1 - f(x)$, which shows that $\text{Kl } f \geq f$.

(c) To prove $\text{Kl}\,(f \vee g) = (\text{Kl } f) \vee (\text{Kl } g)$

Let $f_1 \leq f_2$, for $f_1, f_2 \in I_n^X$.

Then for $g \in I_n^X$, $g\, \bar{\rho}_{n^*}\, f_2 \Rightarrow g\, \bar{\rho}_{n^*}\, f_1$

$\Rightarrow \text{Kl}\,(f2) \geq \text{Kl}\,(f1)$.

Hence, $\text{Kl}\,(f \vee g) \geq (\text{Kl } f) \vee (\text{Kl } g)$

Now assume $((\text{Kl } f) \vee (\text{Kl } g))\,(x) < (\text{Kl}\,(f \vee g))\,(x)$ for some $x \in X$ (3.1)

Then $[\vee\{h \in I_n^X \mid h\, \bar{\rho}_{n^*}\, f\}\,(x)] \wedge [\vee\{h \in I_n^X \mid h\, \bar{\rho}_{n^*}\, g\}\,(x)]$

$> \vee\{h \in I_n^X \mid h\, \bar{\rho}_{n^*}\, (f \vee g)\}\,(x)$

There exist $h_1, h_2 \in I_n^X$ s.t $h_1\, \bar{\rho}_{n^*}\, f$, $h_2\, \bar{\rho}_{n^*}\, g$ and

$h_1(x) > \vee\{h \mid h\, \bar{\rho}_{n^*}\, (f \vee g)\}\,(x)$

$h_2(x) > \vee\{h \mid h\, \bar{\rho}_{n^*}\, (f \vee g)\}\,(x)$

Hence, $(h_1 \wedge h_2)\, \bar{\rho}_{n^*}\, (f \vee g)$ and $(h_1 \wedge h_2)\,(x) > \vee\{h \mid h\, \bar{\rho}_{n^*}\, (f \vee g)\}\,(x)$ (3.2)

(3.2) contradicts (3.1)

Therefore, $\text{Kl}\,(f \vee g) = (\text{Kl } f) \vee (\text{Kl } g)$

(d) To prove Kl (Kl f) = Kl f, it is sufficient to prove g ρ_n* (Kl f) iff g ρ_n* f.

g ρ_n* f $\Rightarrow$ g ρ_n* (Kl f) is always true.

Suppose g ρ_{n*} (Kl f) but g $\bar{\rho}_{n*}$ f, then there exists $\chi_A \in I_n^X$ such that $\chi_A\ \bar{\rho}_{n*}$ g (3.3)

and $(1-\chi_A)\ \bar{\rho}_{n*}$ f. (3.4)

(3.4) $\Rightarrow$ Kl f $\leq \chi_A$

$\Rightarrow$ $\chi_A\ \rho_{n*}$ g (3.5)

(3.3) and (3.5) are contradictory.

Hence, the function f $\rightarrow$ Kl f is a closure operator.

Definition 3.4

The collection $\delta(\rho n^*) = \{f \in I_n^X \mid Kl\,(1-f) = (1-f)\}$ is a fuzzy topology on X and it is called **the n-fuzzy topology induced by ρ_n***.

Proposition 3.5

Let cl be the closure operator w.r.t. to the topology $\delta(E_x(\rho_n^*))$. Then cl f =Kl (^{n}f).

Proof

For $f \in I^X$, cl f $= 1 - \vee\{g \in I^X \mid g\ \overline{E_x\,(\rho_{n*})}\ f\}$

$= 1 - \vee\{g \in I^X \mid {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$ [By Proposition 2.9] (3.6)

Proposition 2.4 (ii),

$g \leq {}^n g \quad \Rightarrow \vee\{g \in I^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$

$\leq \vee\{{}^n g \in I_n^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$ (3.7)

Now $\{{}^n g \in I_n^X / {}^n({}^n g)\ \overline{\rho_{n*}}\ {}^n({}^n f)\} \subseteq \{g \in I^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$

Proposition 2.4 (v),

${}^n({}^n g) = {}^n g \Rightarrow \{{}^n g \in I_n^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\} \subseteq \{g \in I^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$

$\Rightarrow \vee\{{}^n g \in I_n^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\} \leq \vee\{g \in I^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$ (3.8)

(3.7) and (3.8) $\Rightarrow \vee\{g \in I^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\} = \vee\{{}^n g \in I_n^X / {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$

Therefore cl f $= 1 - \vee\{{}^n g \in I_n^X \mid {}^n g\ \overline{\rho_{n*}}\ {}^n f\}$ = Kl (^{n}f)

Theorem 3.6

Any fuzzy open set in $\delta(E_x(\rho_n^*))$ is In-valued and $\delta(E_x(\rho_n^*)) = \delta(\rho_n^*)$.

Proof

Since cl f = Kl (^{n}f), cl f is I_n-valued. $\therefore$ Any fuzzy open set in $\delta(E_x(\rho_{n*}))$ is I_n-valued.

Now $\delta(E_x(\rho_{n*})) = \{f \in I_n^X \mid cl\,(1-f) = 1-f\}$

$= \{f \in I_n^X \mid Kl\,({}^n(1-f)) = 1-f\}$

Proposition 2.8 (i) $\Rightarrow (1-f) = {}^n(1-f)$ for $f \in I_n^X$

Hence, $\delta(E_x(\rho_{n*})) = \{f \in I_n^X \mid Kl\,(1-f) = 1-f\} = \delta(\rho_{n*})$.

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