A Subclass of Meromorphic Starlike Functions with Alternating Coefficients

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ABSTRACT

The aim of the present paper is to introduce a new subclass $\sum_a^*(A,B)$ of meromorphic starlike functions with alternating coefficients in $E=\{z: 0<|z|<1\}$ and investigate coefficients, distortion properties and radius of convexity estimates for the class. Further more it is shown that the class $\sum_a^*(A,B)$ is closed under convex linear combinations, convolutions and integral transforms.

Key words: meromorphic, starlike, convolution and integral transforms.

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1. INTRODUCTION

Let $\sum_s$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $E=\{z: 0<|z|<1\}$ with a simple pole at the origin with residue 1 there. And let $\sum_s$ denote the subclass o A function f(z) in $\sum_s$ is said to be meromorphically starlike of order α if

$$\operatorname{Re}\left\{\frac{-zf'(z)}{f(z)}\right\} > \alpha(z \in E)\ . \tag{1.2}$$

For some α $(0 \leq \alpha < 1)$. We denote by $\sum^*(\alpha)$ the class of all meromorphically starlike functions of order α.

Similarly a function $f(z)$ in $\sum_s$ is said to be meromorphically convex of order α if

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f(z)}\right)\right\} > \alpha(z \in E). \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$ and we denote by $\sum K(\alpha)$ the class of all meromorphically convex functions of order. The class $\Sigma^*(\alpha)$ and similar classes have been extensively studied by Pommerenke [3], Clunie [1], Miller [2] and others.

Let $\sum_{a}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n z^n \ , a_n \geq 0 \qquad (1.4)$$

that are analytic in E .

2. MAIN RESULTS

Definition 2.1: Let $\sum_{a}^{*}(A, B)$ denote the subclass of consisting of functions *f(z)* which satisfy

$$\left|\frac{zf'(z)}{f(z)} + 1\right| < \left|A + B\frac{zf'(z)}{f(z)}\right| \qquad (1.5)$$

For $-1 \leq A < B$, $0 < B \leq 1$ and $z \in E$.

COEFFICIENT ESTIMATES

Theorem 2.1: Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n z^n$ be regular in E. Then *f(z)* is in the class $\sum_{a}^{*}(A, B)$ if and only if $\sum_{n=1}^{\infty} \{(n+1) + (A + Bn)\} a_n \leq B - A$ for $-1 \leq A < B$ and $0 < B \leq 1$.

Proof : Suppose that $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n z^n$, $a_n \geq 0$ is in $\sum_{a}^{*}(A, B)$ then

$$\left| \frac{\frac{zf'(z)}{f(z)} + 1}{A + B\frac{zf'(z)}{f(z)}} \right| = \left| \frac{\sum_{n=1}^{\infty} (-1)^{n-1} (n+1) a_n z^n}{(B-A)\frac{1}{z} - \sum_{n=1}^{\infty} (-1)^{n-1} (A + Bn) a_n z^n} \right| < 1 \quad \text{for all } z \in E. \text{ Since}$$

$Re(z) \leq |z|$ for all z, we have

$$\text{Re} \left\{ \frac{\sum_{n=1}^{\infty} (n+1) a_n z^n}{(B-A)\frac{1}{z} - \sum_{n=1}^{\infty} (A + Bn) a_n z^n} \right\} < 1, \qquad z \in E \qquad (2.7)$$

Now choose the values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.7)

We obtain $\sum_{n=1}^{\infty}\{(n+1)+(A+Bn)\}a_n \le B-A$

Conversely, suppose that (2.6) holds for all admissible values of A and B. We have.

$$H(f,f') = |zf'(z)+f(z)| - |Af(z)+Bzf'(z)| \tag{2.8}$$

$$\left|\sum_{n=1}^{\infty}(-1)^{n-1}(n+1)a_n z^n\right| - \left|(B-A)\frac{1}{z} - \sum_{n=1}^{\infty}(-1)^{n-1}(A+Bn)a_n z^n\right| \quad \text{or}$$

$$|z|\ H(f,f') \le \sum_{n=1}^{\infty}(n+1)\,a_n\,|z|^{n+1} - (B-A) + \sum_{n=1}^{\infty}(A+Bn)\,a_n|z|^{n+1}$$

$$= \sum_{n=1}^{\infty}\{(n+1)+(A+Bn)\}a_n\,|z|^{n+1} \le B-A$$

Since the above inequality holds for all $r = |z|, 0 < r < 1$, letting $r \to 1$

we have $\sum_{n=1}^{\infty}\{(n+1)+(A+Bn)\}a_n \le (B-A)$ by (2.6) Hence it follows that $f(z)$ is in the class $\sum_a^*(A,B)$.

Corollary 2.1 :

If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n$ is in the class $\sum_a^*(A,B)$, then we have

$$a_n \le \frac{B-A}{(n+1)+(A+Bn)}\ , \quad (n \ge 1) \tag{2.9}$$

The result is sharp for the function.

$$f_n(z) = \frac{1}{z} + (-1)^{n-1}\frac{B-A}{(n+1)+(A+Bn)} \tag{2.10}$$

Distortion properties and Radius of convexity

Theorem 2.2 : If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^{n-1}a_n z^n$ is in the class $\sum_a^*(A,B)$,then we have

$$\frac{1}{r} - \frac{B-A}{2+A-B} r \leq f(z) \leq \frac{1}{r} + \frac{B-A}{2+A+B} r \tag{2.11}$$

The result is sharp.

Proof: Suppose that $f(z) \in \sum_a^* (A, B)$. By Theorem 2.1

We have $\sum_{n=1}^{\infty} a_n \leq \frac{B-A}{2+A+B}$

Then $|f(z)| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n z^n \right|$

$$\leq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|$$

$$\leq \frac{1}{r} + \frac{B-A}{2+A+B} r$$

Also, $f(z) \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z| \geq \frac{1}{r} - \frac{(B-A)}{2+A+B} r$

The result is sharp for the function

$$f(z) = \frac{1}{r} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(B-A)}{2+A+B} r$$

Theorem 2.3 : If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n z^n$ is in the class $\sum_a^* (A, B)$ then *f(z)* is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $|z| < r = r(A, B, \delta)$ where

$$r(A, B, \delta) = \underset{n \geq 1}{Inf} \left\{ \frac{(1-\delta)(n+1) + (A+Bm)}{(B-A)n(n+2-\delta)} \right\}^{1/n+1}$$

The result is sharp.

Proof: Let $f(z)$ is in $\sum_{a}^{*}(A, B)$. Then by Theorem 2.1 we have

$$\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n \le 1$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right| \le 1-\delta$

for $|z| \le r(A, B, \delta)$ where $r(A, B, \delta)$ is specified in the statement of the Theorem.

Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=1}^{\infty}(-1)^{n-1} n(n+1) a_n z^{n-1}}{\frac{-1}{z^2}+\sum_{n=1}^{\infty}(-1)^{n-1} n a_n z^{n-1}}\right|$$

$$\le \frac{\sum_{n=1}^{\infty} n\,(n+1) a_n |z|^{n+1}}{1-\sum_{n=1}^{\infty} n\, a_n |z|^{n+1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n\, |z|^{n+1} \le 1 \tag{2.13}$$

By (2.12) it follows that (2.13) is true if

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1} \le \frac{(n+1)+(A+Bn)}{(B-A)}, \quad (n \ge 1).$$

or

$$|z| \le \left\{\frac{(1-\delta)\{(n+1)+(A+Bn)\}}{(B-A)n(n+2-\delta)}\right\}^{1/(n-1)}, (n \ge 1). \tag{2.14}$$

Setting $|z| \le r(A, B, \delta)$ in (2.14), the result follows. The result is sharp for the functions

$$f(z)=\frac{1}{z}+(-1)^{n-1}\frac{(B-A)}{(n+1)+(A+Bn)}z^{n} \qquad (n\geq 1) \qquad (2.15)$$

Convex linear combinations

In this section we shall prove that the class $\sum_{a}^{*}(A,B)$ is closed under linear combinations.

Theorem 2.4: Let $f_0(z)=\frac{1}{z}$ and $f_n(z)=\frac{1}{z}+(-1)^{n-1}\frac{(B-A)}{(n+1)+(A+Bn)}z^{n}, (n\geq 1)$.

Then *f(z)* is in the class $\sum_{a}^{*}(A,B)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z) \text{ where } \lambda_n\geq 0 \quad\text{and}\quad \sum_{n=0}^{\infty}\lambda_n=1$$

Proof: Let $f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)$ with $\lambda_n\geq 0$ and $\sum_{n=0}^{\infty}\lambda_n=1$.

Then

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)=\lambda_0 f_0(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$=\left[1-\sum_{n-1}^{\infty}\lambda_n\right]f_0(Z)+\sum_{n-1}^{\infty}\lambda_n f_n(Z)$$

$$=\left(1-\sum_{n-1}^{\infty}\lambda_n\right)\frac{1}{z}+\sum_{n-1}^{\infty}\lambda_n\left(\frac{1}{z}+(1)^{n-1}\frac{B-A}{(n+1)+(A+Bn)}\right)$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(B-A)}{(n+1)+(A+Bn)}z^{n}$$

Since $\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}\lambda_n\frac{(B-A)}{(n+1)+(A+Bn)}$

By Theorem 2.1, *f(z)* is in the class $\sum_{a}^{*}(A,B)$

Conversely, suppose that the function *f(z)* is in the class

Since

$$a_n \leq \frac{(B-A)}{(n+1)+(A-Bn)}; \qquad n=1,2,3,\ldots\ldots$$

Setting $\lambda_n = \frac{(n+1)+(A+Bn)}{B-A} a_n, \quad (n \geq 1)$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$

It follows that $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$.

This completes the proof of the theorem.

Integral Transforms

In this section we consider integral transforms of the functions in $\sum_{a}^{*}(A,B)$.

Theorem 2.5:

If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^{n-1} a_n z^n$ is in the class $\sum_{a}^{*}(A,B)$ then the integral transforms

$F_c(z) = c\int_0^1 u f(uz)du. (0 < c < \infty)$ are in the class $\sum_{a}^{*}(A,B)$

Proof : Suppose that $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^{n-1} a_n z^n$

is in $\sum_{a}^{*}(A,B)$ Then we have

$$F_c(z) = c\int_0^1 uf(uz)du$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{ca_n}{n+c+1} z^n$$

Since

$$\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} \frac{ca_n}{n+c+1} \leq \frac{(n+1)+(A+Bn)}{B-A} a_n \leq 1.$$

By Theorem 2.1, It follows that $F_c(z)$ is in the class $\sum_a^*(A,B)$.

Remark 2.1: In the above theorems, putting $A = \beta(2\alpha - 1)$ and $B = \beta$ where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we get the results by T. Ram Reddy, P. Thirupathi Reddy and R.B. Sharma [4].

Convolution Properties

Robertson [5] has shown that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in $\sum_z$ then so is their convolution $(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$. We prove the following results for functions in $\sum_a^*(A,B)$

Theorem 2.6: If the functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n b_n z^n$ are in $\sum_a^*(A,B)$ then $(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n b_n z^n$ is in the class $\sum_a^*(A,B)$

Proof: Suppose that $f(z)$ and $g(z)$ are in $\sum_a^*(A,B)$

By Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n \leq 1$$

and $\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} b_n \leq 1$

Since *f(z)* and *g(z)* are regular in *E*, so is *(f *g)(z)*.

Further more,

$$\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n b_n \leq \sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n b_n$$

$$\leq \left[\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n\right]\left[\sum_{n=1}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} b_n\right]$$

$$\leq 1.$$

Hence by Theorem 2.1, *(f *g)(z)* is in the class $\sum_{a}^{*}(A, B)$.

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