Domination Energy of some Well Known Graphs

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ABSTRACT

Representation of a set of vertices in a graph my means of a matrix was introduced by E. Sampath Kumar. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices we can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if, $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S. In the sense that given S we can form the matrix and conversely given the matrix we can draw the graph and identify the set S. here we consider the set S as dominating set and the corresponding matrix is Domination matrix $A_\gamma(G)$ of G , thus the energy $E(G)$ obtained from the Dominating matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$. In this paper we obtain few bounds of $E_\gamma(G)$ and compare the result with $E(G)$.

Key words : Domination Number, Eigen values, Energy of Graph, Adjacency matrix.

AMS Classification: *15A45, 05C50, 05C69.*

1. INTRODUCTION

A subset D of the vertex set V of a graph G is called a dominating set in G, if each vertex of G is either in D or adjacent to at least one vertex in D. A dominating set D in G is minimal dominating set, if no proper subset of D is a dominating set. The minimum cardinality of all the minimal dominating sets is called the domination number and is denoted by $\gamma(G)$.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities, as the heat of formation of a hydrocarbon are related to so called total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is representation of molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent, if there is a bond connecting them.

Eigen values and Eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. From the pioneering work of Coulson [2] there exists a continuous interest towards the general Mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [5,6,7,8].

Representation of a set of vertices in a graph my means of a matrix was introduced by E. Sampath Kumar [4]. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices we can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if, $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S. in the sense that given S we can form the matrix and conversely given the matrix we can draw the graph and identify the set S. here we consider the set S as dominating set and the corresponding matrix is Domination matrix $A_\gamma(G)$ of G, thus the energy $E(G)$ obtained from that matrix is defined as domination energy denoted by, $E_\gamma(G)$ i.e. let the graph G be connected and let its vertices be labeled $v_1, v_2, v_3, \ldots\ldots, v_n$. The domination matrix of G is defined to be the square matrix A_γ (G) corresponding to the dominating set of G. The Eigen values of the dominating matrix are denoted by $\kappa_1, \kappa_2, \kappa_3, \ldots\ldots, \kappa_n$ are said to be A_γ Eigen values of G. since the A_γ matrix is symmetric, its Eigen values are real and can be ordered, $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \ldots\ldots \geq \kappa_n$ $E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|$ - (1). This equation has been chosen so as to be fully analogous to the definition of graph energy [5]-[7] $E = E(G) = \sum_{i=1}^{n} |\lambda_i|$ - (2). Where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \geq \lambda_n$ are the ordinary graph Eigen values [5] that is, the Eigen values of the adjacency matrix $A(G)$. Recall that in the few years, the graph energy $E(G)$ has been extensively studied in the mathematics [9]-[13] and mathematic-chemical literature [14]-[20].

2. MAIN RESULTS

Let $G(V,E)$ be a graph, $S \subseteq V$ & $A(G)$ be the adjacency matrix of G. Replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S.

Definition 1 Minimal Dominating Energy

A dominating set *D* in *G* is minimal dominating set, if no proper subset of *D* is a dominating set. The domination energy $E_\gamma(G)$ obtained for minimal dominating set is called Minimal Dominating Energy denoted by $E_{\gamma-Min}(G)$.

Definition 2 Maximal Dominating Energy

A dominating set *D* in *G* is maximal dominating set, if *D* contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for maximal dominating set is called Maximal Dominating Energy denoted by $E_{\gamma-Max}(G)$.

Definition 3 Dominating Energy

If D is the dominating set whose cardinality is in between minimal and maximal dominating set. Then domination energy $E_\gamma(G)$ obtained for dominating set D is called Dominating Energy denoted by $E_\gamma(G)$.

We approach the study of the domination energy by making the following observations.

Observation 1: If $A(G)$ is the adjacency matrix corresponds to the graph $G(V,E)$ $A_{\gamma-Min}(G)$ is the adjacency matrix corresponding to the minimal dominating set S_{Min} and $A_{\gamma-Max}(G)$ is the adjacency matrix corresponding to the maximum dominating set S_{Max}. Cardinality $|S_{Min}| \le |S| \le |S_{Max}|$ where set S is the dominating set whose cardinality is in between minimal and maximal dominating set.

A graph $G(V,E) \ne K_n$, $n \ge 3$ then $E_{\gamma-Min}(G) \pm \varepsilon \le E_\gamma(G) \le E_{\gamma-Max}(G) \pm \varepsilon$, Where ε is the error factor such that $|\varepsilon| \le 1$

Corollary: A graph $G(V,E) \ne K_n$, $n \ge 3$ then $E(G) \le E_{\gamma-Min}(G)$

Observation 2: A graph $G(V,E) = K_n$, $n \ge 3$ then $E_{\gamma-Min}(G) \pm \varepsilon \ge E_\gamma(G) \ge E_{\gamma-Max}(G) \pm \varepsilon$, Where ε is the error factor such that $|\varepsilon| \le 1$

Corollary: If graph $G(V,E) = K_n$, $n \ge 3$ then $E(G) \ge E_{\gamma-Min}(G)$

Observation 3: $\sum_{i=1}^{n} (\lambda_i)^2 = 2m$. [6]. The domination spectra analog to equation is $\sum_{i=1}^{n} (\kappa_i)^2 = 2m + |S|$, $|S|$ is the cardinality of dominating set for which κ_i is determined. $\therefore \sum_{i=1}^{n} (\kappa_i)^2 > 2m$

3. BOUNDS ON DOMINATION ENERGY

Theorem-1

A graph $G(V,E)$ with $n \ge 3$ and $G \ne K_n$ then $\sqrt{2m + n(n-1)(\det A)^{2/n}} \le E_{\gamma-Min}(G) \le \sqrt{2mn}$ where m is the number of edges and n is the number of vertices

Proof: Let G be a graph with Eigen values $\kappa_1, \kappa_2, \kappa_3, \ldots\ldots, \kappa_n$ corresponding to the minimal dominating set and by the definition of domination energy we have, $E_{\gamma-Min} = E_{\gamma-Min}(G) = \sum_{i=1}^{n} |\kappa_i|$

$$(E_{\gamma-Min})^2 = (E_{\gamma-Min}(G))^2 = \left(\sum_{i=1}^{n} |\kappa_i|\right)^2 = \sum_{i=1}^{n} \kappa_i^2 + 2\sum_{i \ne j} |\kappa_i||\kappa_j|$$

Now using the fact that the arithmetic mean of a set of positive number is greater than or equal to their geometric mean, we have.

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\kappa_i||\kappa_j| \geq \left(\prod_{i\neq j}|\kappa_i||\kappa_j|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n}|\kappa_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} = \left(\prod_{i=1}^{n}|\kappa_i|\right)^{\frac{2}{n}} = (\det A)^{\frac{2}{n}}$$

Using the fact that $\sum_{i=1}^{n}(\kappa_i)^2 > 2m$, we have $(E_{\gamma-Min}(G))^2 \geq 2m + n(n-1)(\det A)^{2/n})$

Therefore lower bound holds

To prove the other inequality, we apply Holders inequality to the two vectors in R^n, $u = (|\kappa_1|, |\kappa_2|, \ldots\ldots, |\kappa_n|)$ and $v = (1,1,\ldots\ldots,1)$, this gives $\sum_{i=1}^{n}|\kappa_i| \leq \sqrt{\sum_{i=1}^{n}\kappa_i^2}\sqrt{n} \geq \sqrt{2mn}$,

$$\sum_{i=1}^{n} k_i^2 > 2m = \sqrt{\sum_{i=1}^{n} k_i^2} > \sqrt{2m} = \sqrt{\sum_{i=1}^{n} k_i^2}\sqrt{n} > \sqrt{2mn}$$

Therefore $\sum_{i=1}^{n}|\kappa_i| \leq \sqrt{2mn}$ this gives the upper bound.

Corollary 1

If $\det A \neq 0$, then $(E_\gamma(G)) \geq \sqrt{2m + n(n-1)} \geq n$.

By another spectral property of a graph G, namely, $\sum_{i<j}\kappa_i\kappa_j = -2m \geq -m$

We have $(E_{\gamma-Min})^2 = (E_{\gamma-Min}(G))^2 = \left(\sum_{i=1}^{n}|\kappa_i|\right)^2 \geq 2m + 2\left|\sum_{i\neq j}|\kappa_i||\kappa_j|\right| \geq 2m + 2|-m| \geq 4m$

Implies $(E_{\gamma-Min}(G)) \geq 2\sqrt{m}$

Corollary 2

If $\det A = 0$, then $(E_{\gamma-Min}(G)) \geq \sqrt{2m}$

Theorem-2

If G is an n-vertex graph without isolated vertices, with $n \geq 3$ and $G \neq K_n$ then $E_{\gamma-Min}(G) \geq 2\sqrt{n} + \varepsilon$.

Proof

Case (a): G is connected

If G is connected graph, then $m \geq n-1$ therefore from above result, $E_{\gamma-Min}(G) \geq 2\sqrt{n-1} \geq 2\sqrt{n}+\varepsilon$.

Case (b): G is disconnected

Let G be composed of $p>1$ components, $G_1, G_2, \ldots, G_p$ having $n_1, n_2, \ldots, n_p$ vertices, respectively $n_1, n_2, \ldots, n_p = n$ .since G has no isolated vertices, for all values of j it must be $n_j \geq 2$ and therefore $\sqrt{n_j - 1} \geq 1$, $E_{\gamma-Min}(G) = E_{\gamma-Min}(G_1) + E_{\gamma-Min}(G_2) + \ldots\ldots + E_{\gamma-Min}(G_p)$ and $E_{\gamma-Min}(G_j) \geq 2\sqrt{n_j - 1}$ for $j = 1,2,\ldots,p$, consequently

$$E_{\gamma-Min}(G) \geq 2\left(\sqrt{n_1-1} + \sqrt{n_2-1} + \ldots\ldots\sqrt{n_p-1}\right) = 2\sqrt{\left(\sqrt{n_1-1} + \sqrt{n_2-1} + \ldots\ldots\sqrt{n_p-1}\right)^2}$$

$$= 2\sqrt{\left((n_1-1)+(n_2-1)\right) + \ldots + (n_p-1) + 2\sum_{j<k}\sqrt{n_j-1}\sqrt{n_k-1}}$$

$\geq 2\sqrt{n-p+p(p-1)} = 2\sqrt{n-1+(p-1)^2}$ because there are P (p-1)/2 summands of the form $\sqrt{n_j-1}\sqrt{n_k-1}$, each being greater than or equal to unity.

Thus the graph with $p \geq 1$, and without isolated vertices $E_{\gamma-Min}(G) \geq 2\sqrt{n-1} \geq 2\sqrt{n}+\varepsilon$

Theorem-3

$K_{n,n}$ is a Complete regular bipartite graph with $n \geq 3$, then $E_{\gamma-Min}(K_{n,n}) \leq 2|V| - 2 \leq 2n-2$ where $|V|$ is the cardinality of vertices in G.

Proof: The Proof uses the spectral property of a graph, that is $\sum_{i=1}^{n} \kappa_n^{\ 2} > 2m$, Cauchy-Schwarz inequality for the vectors $u = (|\kappa_1|, |\kappa_2|, \ldots\ldots, |\kappa_n|)$ and $v = (1,1,\ldots\ldots,1)$ in R^{n-2} and thus

$$(\kappa_1 + \kappa_1 +, \ldots\ldots, \kappa_n)^2 \leq (\kappa^2{}_1 + \kappa^2{}_1 +, \ldots\ldots, + \kappa^2{}_n)(1 + 1 +, \ldots\ldots\ldots\ldots, +1)$$

$$\sum_{i=1}^{n} |\kappa_n| \leq \sqrt{(\kappa_1^{\ 2})n} \leq \sqrt{2mn} \leq 2n-2$$

or we can prove the result analogies to the theory of ordinary graph energy.

$$\sum_{i=1}^{n} \kappa_n^{\ 2} \leq \sqrt{(n-2)(2m-2\kappa_1^{\ 2})}$$

and searching for the maximum value of $f(x)$ at the appropriate value of x we get the bound

$$E_{\gamma-Min}(K_{1,n}) \leq \frac{n}{2\sqrt{2}}\left(\sqrt{n}+\sqrt{2}\right) \leq 2n-2 = 2|V|-2.$$

Theorem-4. [9], [10], [16], [17]

A graph $G(V,E)$ with $n \geq 3$ then $E_{\gamma-Min}(G) \leq \frac{n}{2}\left(\sqrt{n}+1\right)+\varepsilon$ where m is the number of edges and n is the number of vertices

Proof: similar to the proof in. [9], [10], [16], [17]

Theorem-5

A graph $G(V,E)$ is a complete graph with $n \geq 3$ then $E_{\gamma}(K_n) \leq \sqrt{mn}$ where m is the number of edges and n is the number of vertices

Proof: similar to the proof of theorem 1 the upper bound holds of the complete graph and from the observation 2 and corollary $E_{\gamma-Min}(G) \pm \varepsilon \geq E_{\gamma}(G) \geq E_{\gamma-Max}(G) \pm \varepsilon$, Where ε is the error factor such that $|\varepsilon| \leq 1$, we have

$E(G) \geq E_{\gamma-Min}(G)$, $E(G) = 2(n-1)$ [7]

$2(n-1) \geq E_{\gamma-Min}(G)$, $E_{\gamma-Min}(G) \leq 2(n-1) \leq \sqrt{mn}$

Theorem-6

A graph $G(V,E)$ with $n \geq 3$ then $E_{\gamma-Min}(K_{1,n-1}) \leq E_{\gamma-Min}(T_n) \leq E_{\gamma-Min}(P_n)$ where $K_{1,n-1}$ is star graph with n vertices T_n is tree with n vertices and P_n is path with n vertices.

Proof: we have $E(K_{1,n-1}) \leq E(T_n) \leq E(P_n)$ hence we can prove the above result analogies to the theory of ordinary graph energy [7].

4. DISCUSSION

At this time it is difficult to see how good the estimates given in the bounds are. As concluding remark we mention that the inequalities stated here have analogies in the theory of ordinary graph energy.

5. OPEN PROBLEM.

1. The relation between these parameters can be extended to other classes of graphs and other types of Domination.
2. Similar bounds i.e. relation between Domination energy, Energy of graph could be extended to sub graphs, partitioning of graphs, distance energy and other related topics.
3. Criticalness of domination energy can be analyzed.

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