On Some Functional Equations

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Abstract

In this paper another proof has been given for a result proved by F. Vajzović and later by K. Lajkó.

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1 Preliminaries

We start off with trigonometric functional equations. We are familiar with the formula

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y, \quad \text{for } x, y \in \mathbb{R}, \tag{1.1}$$
$$\cos(x-y) - \cos(x+y) = 2\sin x \sin y, \quad \text{for } x, y \in \mathbb{R}, \tag{1.2}$$

which leads to the study of

$$f(x+y) + f(x-y) = 2\sin x \cos y, \quad \text{for } x, y \in \mathbb{R}, \tag{1.3}$$
$$f(x+y) - f(x-y) = 2\sin x \sin y, \quad \text{for } x, y \in \mathbb{R}. \tag{1.4}$$

Result 1.1 *A function $f : \mathbb{R} \to \mathbb{R}$ satisfies* (1.3) *if and only if* $f(x) = \sin x$.

Proof: Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.3). Then

$$f(x+y) + f(x-y) = \sin(x+y) + \sin(x-y)$$

or

$$f(x+y) - \sin(x+y) = \sin(x-y) - f(x-y)$$

which with $y = 0$ implies

$$f(x) - \sin x = \sin x - f(x)$$

or

$$f(x) = \sin x,$$

which is to be proved. Evidently $f(x) = \sin x$ satisfies (1.3).

Result 1.2 *Whereas $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$f(x+y) + f(x-y) = 2\sin x \sin y, \quad \text{for } x, y \in \mathbb{R} \tag{1.5}$$

has no solutions, f satisfying (1.4) *has solution*

$$f(x) = -\cos x + c.$$

Proof: Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.5). Then $y = 0$ in (1.5) gives $f(x) = 0$ which is not a solution of (1.5). But if we consider $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.4), then

$$f(x+y) - f(x-y) = \cos(x-y) - \cos(x+y), \quad \text{for } x, y \in \mathbb{R},$$

which is

$$f(x+y) + \cos(x+y) = f(x-y) + \cos(x-y),$$

that is,

$$\begin{aligned} f(u) + \cos u &= f(v) + \cos v, \quad \text{for } u, v \in \mathbb{R} \\ &= c \end{aligned}$$

(put $x + y = u,\ x - y = v$). This proves the result.

No surprises here. Now we consider the generalization of (1.3) and (1.4), namely,

$$f(x+y) + g(x-y) = 2\sin x \cos y, \quad \text{for } x, y \in \mathbb{R}, \tag{1.6}$$
$$f(x+y) + g(x-y) = 2\sin x \sin y, \quad \text{for } x, y \in \mathbb{R}. \tag{1.7}$$

Result 1.3 *The only solution $f, g : \mathbb{R} \to \mathbb{R}$ of (1.6) is given by*

$$f(x) = \sin x + c, \quad g(x) = -\sin x - c, \quad \text{for } x \in \mathbb{R}.$$

Proof: With $x = 0$ (1.6) becomes

$$f(y) + g(-y) = 0, \quad \text{for } y \in \mathbb{R}. \tag{1.8}$$

Then (1.6) becomes

$$\begin{aligned} f(x+y) - f(y-x) &= 2\sin x \cos y, \quad \text{for } x, y \in \mathbb{R} \\ &= \sin(x+y) + \sin(x-y), \end{aligned}$$

which is

$$f(x+y) - \sin(x+y) = f(y-x) - \sin(y-x),$$

which results to

$$f(x) = \sin x + c, \quad g(x) = -\sin x - c.$$

Result 1.4 *Now we will solve (1.7) for $x, y \in \mathbb{R}$ and prove*

$$f(x) = \cos x + c, \quad g(x) = -\cos x - c.$$

Proof: With $y = 0$, (1.7) yields

$$f(x) + g(x) = 0, \quad \text{for } x \in \mathbb{R}, \tag{1.9}$$

then (1.7) becomes

$$\begin{aligned} f(x+y) - f(x-y) &= 2\sin x \sin y \\ &= \cos(x+y) - \cos(x-y), \end{aligned}$$

that is,

$$f(x+y) - \cos(x+y) = f(x-y) - \cos(x-y),$$

which results to

$$f(x) = \cos x + c, \quad g(x) = -\cos x - c.$$

All these results are obtained without any regularity condition on f and g. Now we treat differentiable functions and prove two results:

$$f(x+y)f(x-y) = f'(x)^2 - f(y)^2, \quad \text{for } x, y \in \mathbb{R}, \tag{1.10}$$
$$f(x+y)f(x-y) = f(x)^2 - f'(y)^2, \quad \text{for } x, y \in \mathbb{R}. \tag{1.11}$$

First we will take up (1.10).

Result 1.5 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be differentiable twice and satisfy* (1.10). *Then* $f = 0$ *is the only solution of* (1.10).

Proof: Letting $y = 0$ in (1.10) to get

$$f(x)^2 = f'(x)^2 - c \tag{1.12}$$

so that (1.10) can be rewritten as

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2 + c, \tag{1.13}$$

where $c = f(0)^2$.

Put $x = 0$ in (1.13) to get

$$f(y)f(-y) = -f(y)^2,$$

that is,

$$f(y) = 0 \quad \text{or} \quad f(y) = -f(-y).$$

Differentiating first with respect to y and then with respect to x, we get

$$f''(x+y)f(x-y) = f(x+y)f''(x-y)$$

or

$$f''(u)f(v) = f(u)f''(v) \qquad (x+y=u,\ x-y=v)$$

that is,

$$f''(u) = kf(u).$$

The solutions of this differential equation are

$$f(x) = cx \quad \text{or} \quad c\sin bx \quad \text{or} \quad c\sin hbx.$$

Since (1.12) holds, none of them are solutions of $f(x)$ $(c \neq 0)$. Thus $f(x) = 0$ is the only solution of (1.10).

Now we take up (1.11).

Result 1.6 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be twice differentiable and satisfy* (1.11). *Then* $f = 0$ *is the only solution of* (1.11).

Proof: Of course $f = 0$ is a solution of (1.11). With $x = 0$, (1.11) gives

$$f(y)f(-y) = d - f'(y)^2 \tag{1.14}$$

so that (1.11) becomes

$$f(x+y)f(x-y) = f(x)^2 + f(y)f(-y) - d,$$

which differentiating first with respect to x and then with respect to y, we get as before

$$f''(x+y)f(x-y) = f(x+y)f''(x-y),$$

that is,

$$f''(u) = kf(u).$$

The solutions are $f(x) = cx$ or $c \sin bx$ or $c \sin hbx$. None of them can be solutions which can be seen from (1.14) ($c \neq 0$). Thus $f = 0$ is the only solution.

The functional equation

$$f(x+y)f(x-y) = f^2(x) - f^2(y), \quad \text{for } x, y \in \mathbb{R}$$

is the source for (1.10) and (1.11) and the continuous solutions are given by $f(x) = cx$ or $c \sin bx$ or $c \sin hbx$.

2 Main Result

First F. Vajzović [3] and later K. Lajkó [2] have proved the following theorem:

Theorem 2.1 *If* $f : \mathbb{R}^+ \to \mathbb{R}$ *is a measurable solution of the functional equation*

$$2f\left(\frac{t}{2}\right) = f\left(\frac{tx}{1+x}\right) + f\left(\frac{t}{1+x}\right) - f\left(\frac{x}{1+x}\right) - f\left(\frac{1}{1+x}\right), \tag{2.1}$$

for $t, x \in \mathbb{R}^+$, *where* $\mathbb{R}$ *is the set of real numbers and* $\mathbb{R}^+$ *the set of all positive reals, then* f *has the form*

$$f(x) = b \log 2x + a\left(x - \frac{1}{2}\right), \quad x \in \mathbb{R}^+ \tag{2.2}$$

where a and b are arbitrary constants.

Here we are giving yet another simple proof of the above theorem having some common intersection with the method in [2].

Proof: Replacing t by tu in (2.1), we get

$$2f\left(\frac{tu}{2}\right) = f\left(\frac{tux}{1+x}\right) + f\left(\frac{tu}{1+x}\right) - f\left(\frac{x}{1+x}\right) - f\left(\frac{1}{1+x}\right). \qquad (2.3)$$

By defining

$$g_\lambda(x) = f(\lambda x) - f(x), \quad \text{for } \lambda, x \in \mathbb{R}^+, \qquad (2.4)$$

from (2.1) and (2.3), we obtain

$$2g_u\left(\frac{t}{2}\right) = g_u\left(\frac{tx}{1+x}\right) + g_u\left(\frac{t}{1+x}\right),$$

which by the substitutions $\frac{tx}{1+x} = y$ and $\frac{t}{1+x} = z$ reduces to

$$g_u(y) + g_u(z) = 2g_u\left(\frac{y+z}{2}\right), \quad \text{for } y, z \in \mathbb{R}^+. \qquad (2.5)$$

This is Jensen. Since g_u given by (1.4) is measurable, we have [1],

$$g_u(x) = a(u)x + b(u), \quad \text{for } x \in \mathbb{R}^+$$

and thus from (2.4), we obtain

$$f(\lambda x) = f(x) + a(\lambda)x + b(\lambda), \quad \text{for } \lambda, x \in \mathbb{R}^+. \qquad (2.6)$$

For $x = 1$, (2.6) yields

$$f(\lambda) = f(1) + a(\lambda) + b(\lambda). \qquad (2.7)$$

Now (2.6) and (2.7) imply

$$a(\lambda x) + b(\lambda x) - a(x) - b(x) - a(\lambda)x - b(\lambda) = 0. \qquad (2.8)$$

Interchanging x and λ in (2.8), we have

$$a(\lambda x) + b(\lambda x) - a(\lambda) - b(\lambda) - a(x)\lambda - b(x) = 0. \qquad (2.9)$$

Use (2.8) and (2.9) to get

$$a(\lambda)(x-1) = a(x)(\lambda - 1),$$

that is,

$$a(\lambda) = a(\lambda - 1), \tag{2.10}$$

where a is a constant.

Now (2.9) in (2.8) gives

$$b(\lambda x) = b(\lambda) + b(x), \quad \lambda, x \in \mathbb{R}^+. \tag{2.11}$$

This is logarithmic and since by (2.11) b is measurable, b is given by [1]

$$b(x) = b \log x, \tag{2.12}$$

where b is a constant.

By (2.7), (2.11) and (2.12) we obtain

$$f(x) = f(1) + a(x - 1) + b \log x, \quad x \in \mathbb{R}^+. \tag{2.13}$$

Since $f(\frac{1}{2}) = 0$ from (2.1), $x = \frac{1}{2}$ in (2.13) gives

$$f(1) - \frac{a}{2} + b \log 2 = 0, \tag{2.14}$$

so that (2.13) and (2.14) show that f indeed has the form given by (2.2).

References

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