(λ, μ) - Fuzzy Subnear - Rings and (λ, μ) - Fuzzy Ideals of Near - Rings

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Abstract

In this paper, we introduce the notions of (λ, μ) -fuzzy subnear-ring and (λ, μ) -fuzzy ideal of near-rings and find more generalized concepts than those introduced by others. The characterization of such (λ, μ) -fuzzy ideals are also obtained.

Key words : (λ, μ) -fuzzy subnear-ring, $(\lambda, /)$ -fuzzy ideal..

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1. Introduction

The notion of a fuzzy set was introduced by Zadeh [7] in 1965. Rosenfeld [3] introduced the notion of fuzzy subgroup in 1971. The notions of fuzzy subnear-ring and fuzzy ideal of near-rings were introduced by Salah Abou-Zaid [4]. In particular, Bhakat and Das introduced the concept of $(\in, \in\vee q)$ - fuzzy subgroups [1], $(\in, \in\vee q)$ - fuzzy subrings and $(\in, \in\vee q)$ -fuzzy ideals [2]. Yuan et al. [6] introduced the concept of fuzzy subgroup with thresholds of a group (also called a (λ, μ) -fuzzy subgroup). Yao introduced the notions of (λ, μ) -fuzzy subrings and (λ, μ) -fuzzy ideals of rings [5] which can be regarded as a generalization of Bhakat and Das's correspondence concepts.

In this paper, the notions of (λ, μ) -fuzzy subnear-ring and (λ, μ) -fuzzy ideal of nearrings are introduced and find more generalized concepts than those introduced by others. We also obtained the characterization of such (λ, μ) -fuzzy ideals.

2. Preliminaries

In this section, we shall present basic definitions and results required in the sequel. By a near-ring we mean a non-empty set N with two binary operations "+" and "•" satisfying the following axioms:

(i) $(N, +)$ is a group,

(ii) $(N, \bullet)$ is a semigroup,

(iii) $(x+y) \bullet z = x \bullet z + y \bullet z$ for all $x, y, z \in N$.

If P and Q are two non-empty subsets of N, we define

$PQ = \{ab \mid a \in P, b \in Q\}$ and $P * Q = \{a(b+i) - ab \mid a, b \in P, i \in Q\}$.

A subgroup M of a near-ring N is called a subnear-ring of N if $MM \subseteq M$.

A subset I of a near-ring N is called an ideal of N if

(i) $(I, +)$ is a normal subgroup of $(N, +)$,

(ii) $IN \subseteq I$,

(iii) $a(b+i) - ab \in I$ for all $a, b \in N$ and $i \in I$, that is, $N * I \subseteq I$.

A normal subgroup R of $(N, +)$ with (ii) is called a right ideal of N while a normal subgroup L of $(N, +)$ with (iii) is called a left ideal of N.

We now review some fuzzy logic concepts.

A function A from a nonempty set X to the unit interval $[0, 1]$ is called a fuzzy subset of X [7].

A map f from a near-ring N_1 into a near-ring N_2 is called a homomorphism if $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in N_1$.

Let f be any function from a set S into a set T, A be any fuzzy subset of S and B be any fuzzy subset of T. The image of A under f, denoted by $f(A)$, is a fuzzy subset of T defined by

$$(f(A))(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) \text{ if } f^{-1}(y) = \phi \\ 0 \quad \text{otherwise, where } y \in T. \end{cases}$$

The preimage of B under f, symbolized by $f^{-1}(B)$, is a fuzzy subset of S defined by $(f^{-1}(B))(x) = B(f(x))$ for all $x \in S$.

Throughout this paper, N will denote a near-ring unless otherwise specified. We denote by K_I the characteristic function of a subset I of N. The characteristic function of N is denoted by **N**, that is, **N** : $N \rightarrow [0, 1]$ mapping every element of N to 1.

Definition 2.1 [4]

A fuzzy subset A of N is said to be a fuzzy subnear-ring of N if for all $x, y \in N$,

(i) $A(x-y) \geq \min\{A(x), A(y)\}$,

(ii) $A(xy) \geq \min\{A(x), A(y)\}$.

Definition 2.2 [4]

A fuzzy subset A of N is said to be a fuzzy ideal of N if

(i) A is a fuzzy subnear-ring of N,

(ii) $A(y + x - y) \geq A(x)$ for all $x, y \in N$,

(iii) $A(xy) \geq A(x)$ for all $x, y \in N$,

(iv) $A(a(b + i) - ab) \geq A(i)$ for all $a, b, i \in N$.

A fuzzy subset with (i), (ii) and (iii) is called a fuzzy right ideal of N whereas a fuzzy subset with (i), (ii) and (iv) is called a fuzzy left ideal of N.

Definition 2.3

A fuzzy subset A of N is said to be an $(\in, \in \vee q)$ -fuzzy subnear-ring of N if for all $x, y \in N$,

(i) $A(x + y) \geq \min\{A(x), A(y), 0.5\}$,

(ii) $A(-x) \geq \min\{A(x), 0.5\}$,

(iii) $A(xy) \geq \min\{A(x), A(y), 0.5\}$.

Definition 2.4

A fuzzy subset A of N is said to be an $(\in, \in \vee q)$ -fuzzy ideal of N if

(i) A is an $(\in, \in \vee q)$ -fuzzy subnear-ring of N,

(ii) $A(y + x - y) \geq \min\{A(x), 0.5\}$ for all $x, y \in N$,

(iii) $A(xy) \geq \min\{A(x), 0.5\}$ for all $x, y \in N$,

(iv) $A(x(y + i) - xy) \geq \min\{A(i), 0.5\}$ for all $x, y, i \in N$.

Definition 2.5. [6]

Let $\lambda, \mu \in [0, 1]$ and $\lambda < \mu$. Let A be a fuzzy subset of a group G. Then A is called a fuzzy subgroup with thresholds of G or a (λ, μ) -fuzzy subgroup of G if for all $x, y \in G$,

(i) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,

(ii) $A(x^{-1}) \vee \lambda \geq A(x) \wedge \mu$.

3. (λ, μ) -fuzzy subnear-rings and (λ, μ) -fuzzy ideals

Based on the concepts of fuzzy subgroups with thresholds were introduced by Yuan [6], (λ, μ) -fuzzy subrings and (λ, μ) -fuzzy ideals were introduced by Yao [5], we introduce the following concept. In the following discussion, we always assume that $0 \leq \lambda < \mu \leq 1$.

Definition 3.1

A fuzzy subset A of N is said to be a (λ, μ) -fuzzy subnear-ring of N if for all $x, y \in N$,

(i) $A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,

(ii) $A(-x) \vee \lambda \geq A(x) \wedge \mu$,

(iii) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$.

Theorem 3.2

A fuzzy subset A of N is said to be a (λ, μ) -fuzzy subnear-ring of N if and only if for all x, y $\in N$,

(i) $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,

(ii) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$.

Remark 3.3

A fuzzy subnear-ring is a (λ, μ) -fuzzy subnear-ring with $\lambda = 0$ and $\mu = 1$, and a $(\in, \in \vee q)$ - fuzzy subnear-ring is a (λ, μ) -fuzzy subnear-ring with $\lambda = 0$ and $\mu = 0.5$.

Definition 3.4

A fuzzy subset A of N is said to be a (λ, μ) -fuzzy ideal of N if

(i) A is a (λ, μ) -fuzzy subnear-ring of N,

(ii) $A(y+x-y) \vee \lambda \geq A(x) \wedge \mu$ for all $x, y \in N$,

(iii) $A(xy) \vee \lambda \geq A(x) \wedge \mu$ for all $x, y \in N$,

(iv) $A(a(b+i)-ab) \vee \lambda \geq A(i) \wedge \mu$ for all $a, b, i \in N$.

A fuzzy subset with (i), (ii) and (iii) is called a (λ, μ) -fuzzy right ideal of N whereas a fuzzy subset with (i), (ii) and (iv) is called a (λ, μ) -fuzzy left ideal of N.

Remark 3.5

A fuzzy ideal is a (0, 1) -fuzzy ideal and a (, $\vee q$) -fuzzy ideal is a (0, 0.5) -fuzzy ideal.

Theorem 3.6

Let $\{Ai\ ;\ i \in J\}$ be any family of (λ, μ) -fuzzy subnear-rings (ideals) of N. Then $A = \cap Ai$ is a (λ, μ) -fuzzy subnear-ring (ideal) of N.

Theorem 3.7

A non-empty subset I of N is a subnear-ring (ideal) of N if and only if K_I is a (λ, μ) - fuzzy subnear-ring (ideal) of N.

Proof :

We prove the result for ideals. Let I be an ideal of N. It is clear that K_I is a fuzzy ideal of N. By Remark 3.5, K_I is a (λ, μ) -fuzzy ideal of N. Conversely, let K_I be a (λ, μ) -fuzzy ideal of N. For any $x, y \in I$, we have $K_I(x - y) \vee \lambda \geq K_I(x) \wedge K_I(y) \wedge \mu \geq 1 \wedge 1 \wedge \mu = \mu$, and so $K_I(x - y) = 1$. Thus $x - y \in I$.

Let $a \in N$ and $x \in I$. Then $K_I(a + x - a) \vee \lambda \geq K_I(x) \wedge \mu \geq 1 \wedge \mu = \mu$, and thus $K_I(a + x - a) = 1$. This shows that $a + x - a \in I$, and therefore $(I, +)$ is a normal subgroup of $(N, +)$.

Now let $a \in N$ and $x \in I$. Then $K_I(xa) \vee \lambda \geq K_I(x) \wedge \mu \geq 1 \wedge \mu = \mu$, and so $xa \in I$. Finally, let $a, b \in N$ and $i \in I$. Then

$K_I(a(b + i) - ab) \vee \lambda \geq K_I(i) \wedge \mu \geq 1 \wedge \mu = \mu$, which implies that $a(b + i) - ab \in I$. Consequently, I is an ideal of N.

Theorem 3.8

Let $f : N_1 \rightarrow N_2$ be an onto homomorphism of near-rings and let A be a (λ, μ) -fuzzy subnear-ring (ideal) of N_1. Then $f(A)$ is a (λ, μ) -fuzzy subnear- ring (ideal) of N_2.

Proof

We prove the result in the case of (λ, μ) -fuzzy ideal.

For all $y_1, y_2 \in N_2$, we have

$$\begin{aligned}
f(A)(y_1 - y_2) \vee \lambda &= \sup\{A(x_1 - x_2) | f(x_1 - x_2) = y_1 - y_2\} \vee \lambda \\
&= \sup\{A(x_1 - x_2) \vee \lambda | f(x_1 - x_2) = y_1 - y_2\} \\
&\geq \sup\{A(x_1) \wedge A(x_2) \wedge \mu | f(x1) = y_1, f(x_2) = y_2\} \\
&= \sup\{A(x_1) \wedge | f(x_1) = y_1\} \wedge \sup\{A(x_2) | f(x_2) = y_2\} \wedge \mu \\
&= f(A)(y_1) \wedge f(A)(y_2) \wedge \mu,
\end{aligned}$$

and

$$\begin{aligned}
f(A)(y_1 y_2) \vee \lambda &= \sup\{A(x_1 x_2) | f(x_1 x_2) = y_1 y_2\} \vee \lambda \\
&= \sup\{A(x_1 x_2) \vee \lambda | f(x_1 x_2) = y_1 y_2\} \\
&\geq \sup\{A(x_1) \wedge A(x_2) \wedge \mu | f(x_1) = y_1, f(x_2) = y_2\} \\
&= \sup\{A(x_1) \wedge | f(x_1) = y_1\} \wedge \sup\{A(x_2) | f(x_2) = y_2\} \wedge \mu \\
&= f(A)(y_1) \wedge f(A)(y_2) \wedge \mu.
\end{aligned}$$

Similarly, we have

$f(A)(y_2 + y_1 - y_2) \vee \lambda \geq f(A)(y_1) \wedge \mu$,

$f(A)(y_1 y_3) \vee \lambda \geq f(A)(y_1) \wedge \mu$,

$f(A)(y_1(y_2 + y_4) - y_1 y_2) \vee \lambda \geq f(A)(y_1) \wedge \mu$ for all $y_1, y_2, y_3, y_4 \in N_2$.

Therefore, $f(A)$ is a (λ, μ) -fuzzy ideal of N_2.

Theorem 3.9

Let $f: N_1 \rightarrow N_2$ be a homomorphism of near-rings and let B be a (λ, μ) -fuzzy subnear-ring (ideal) of N_2. Then $f^{-1}(B)$ is a (λ, μ) -fuzzy subnear- ring (ideal) of N_1.

Proof

We prove the result in the case of (λ, μ) -fuzzy ideal.

For all $x_1, x_2 \in N_1$, we have

$$\begin{aligned} f^{-}1(B)(x_1 - x_2) \vee \lambda &= B(f(x_1 - x_2)) \vee \lambda \\ &= B(f(x_1) - f(x_2)) \vee \lambda \\ &\geq B(f(x_1)) \wedge B(f(x_2)) \wedge \mu \\ &= f^{-1}(B)(x_1) \wedge f^{-1}(B)(x_2) \wedge \mu, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(B)(x_1 x_2) \vee \lambda &= B(f(x_1 x_2)) \vee \lambda \\ &= B(f(x_1) f(x_2)) \vee \lambda \\ &\geq B(f(x_1)) \wedge B(f(x_2)) \wedge \mu \\ &= f^{-1}(B)(x_1) \wedge f^{-1}(B)(x_2) \wedge \mu. \end{aligned}$$

Similarly, we have

$f^{-1}(B)(x_2 + x_1 - x_2) \vee \lambda = f^{-1}(B)(x_1) \wedge \mu$,

$f^{-1}(B)(x_1 x_3) \vee \lambda = f^{-1}(B)(x_1) \wedge \mu$,

$f^{-1}(B)(x_1(x_2 + x_4) - x_1 x_2) \vee \lambda = f^{-1}(B)(x_4) \wedge \mu$ for all $x_1, x_2, x_3, x_4 \in N_1$.

Therefore, $f^{-1}(B)$ is a (λ, μ) -fuzzy ideal of N_1.

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