TOPOLOGICAL INDICES OF SOME GRAPH OPERATIONS

[1]Maru U [2]Arockiaraj PS and [3]James Albert A

Abstract

The Wiener index for a connected graph G is defined as $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v)$, where the summation is taken over all unordered pair of vertices of $V(G)$. The n-Steiner Wiener index of a connected graph G is $W_n(G) = \sum_{S\subseteq V(G)} d(S)$, where $d(S)$ is the Steiner distance of the n-element subset S of $V(G)$ and the summation is taken over all unordered n-element subsets of $V(G)$. The first Zagreb index $M_1(G)$ is defined as $M_1(G) = \sum_{v\in V}[\deg(v)]^2$. In this paper, the Wiener index and the first Zagreb index of neighbourhood Corona of two graphs, Wiener index for Splitting graph and 3-Steiner Wiener index of the Complementary Prism, edge joining of two graphs and duplicating graph are found.

Keywords: Wiener index, First Zagreb index, Steiner Wiener index, neighbourhood corona, splitting graph, complementary prism.

AMS Subject Classification Number. 05C12.

1 Introduction

All the graphs considered in this paper are finite, undirected and simple. We refer the reader to [6] for terminology and notations. A *graph* $G = (V,E)$ is a set of finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices*, while u and e are incident, as are v and e.

[1]Research Scholar, Karpagam University, Coimbatore and Faculty in the Department of Mathematics, Nirmala College for Women, Coimbatore - 641 018, Tamil Nadu, India.

[2]Department of Mathematics, Mepco Schlenk Engineering College, Sivakasi - 626005, Tamil Nadu, India

[3]Department of Science and Humanities, Hindustan College of Engineering and Technology, Coimbatore - 641032, Tamil Nadu, India

The *degree of a vertex v* in a graph G is the number of edges of G incident with v, which is denoted by $\deg_G(v)$ or simply by $\deg(v)$. A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is an *end vertex* of G

The *distance* $d_G(u,v)$ from a vertex u to a vertex v in a connected graph G, or simply $d(u, v)$ is the length of the shortest $u-v$ path in G [1]. A $u-v$ path of length $d(u, v)$ is called a $u-v$ *geodesic*.

The Wiener index is the first and most studied topological index, both from theoretical point and applications. The *Wiener number* or *Wiener index* $W(G)$ of a graph G was put forward in 1974 by Harold Wiener [11]. Its applications in the modeling of various physio–chemicals, biological and pharmacological properties of organic molecules are outlined in several monographs and reviews. The Wiener index $W(G)$ of a graph G is defined to be $W(G)=\sum_{i<j} d(v_i,v_j)$.

The Wiener index also be defined by considering the distance matrix of a graph G denoted by $D(G)$ and the $(i,j)^{th}$ entry in $D(G)$ is equal to $d(v_i,v_j)$ [3,4]. So the sum of the elements of i^{th} row of $D(G)$ is equal to $\sum_{j=1}^{n} d(v_i,v_j)$, where n is the number of vertices in G.

The *distance of a vertex u* of a graph G denoted by $d(u|G)$ and is defined as $d(u\,|\,G) = \sum_{v\in V(G)} d(u,v)$..

From this, the Wiener index of a graph G can also be defined as $W(G)=\frac{1}{2}\sum_{u\in V(G)} d(u\,|\,G)$.

The Wiener index can be calculated for some particular classes of graphs. But as such there is no exact formula for finding the Wiener index of a general graph.

The *Steiner distance* of the set S of vertices in a connected graph G, $d_G(S)$ is the number of edges in a smallest connected subgraph of G contains S and such a connected subgraph is called as a *Steiner tree* for S. If $|S| = 2$, then the Steiner distance of S is the distance between two vertices of S. Further if $S = \{u, v\}$, then $d_G(S)=d(u,v)$ while if $|S| = n$, then $d_G(S)=n-1$. Steiner trees have applications to multiprocessor networks. For example, it may be desired to connect a certain set of processors with a sub network that uses the fewest communication links. A Steiner tree for the vertices that need to be connected corresponds to such a sub network.

The *n-Steiner Wiener index* of a connected graph G is $W_n(G) = \sum_{S \subseteq V(G)} d_G(S)$, where $d_G(S)$ is the Steiner distance of the n-element subset S of $V(G)$ and the summation is taken over all unordered n-element subsets of $V(G)$. In other words, the n-Steiner Wiener index of a connected graph G is $W_n(G) = \frac{d_n(G)}{\binom{p}{n}}$ where $d_n(G) = \sum \{d_G(S) \ / \ S \subseteq V(G), \ |S| = n\}$.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestic [6]. It is an important molecular descriptor and has been closely correlated with many chemical properties [6, 8]. The *first Zagreb index* $M_1(G)$ is defined as $M_1(G) = \sum_{v \in V(G)} [\deg_G(v)]^2$ [6].

Let G_1 and G_2 be two graphs having n_1 and n_2 vertices and m_1 and m_2 edges respectively. Then the *neighbourhood corona* $G_1 * G_2$ is the graph obtained by taking n_1 copies of G_2 and each member of neighbours of every vertex v of G_1 is adjacent to all the vertices of the copy of G_2 corresponding to v.

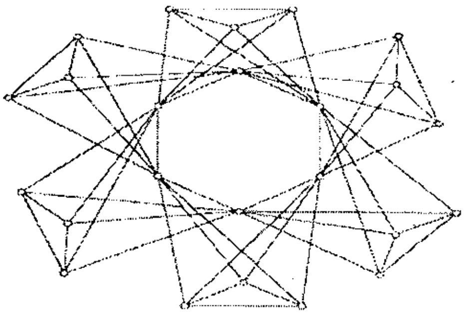

Figure 1.$C_6 * K_3$

Splitting graph $S(G)$ was introduced by Sampath Kumar and Walikar [10]. For each vertex v of a graph G, take new vertex v' and join v' to all vertices of G adjacent to v. The graph $S(G)$ thus obtained is called the *splitting graph* of G.

The *complementary prism* $G\overline{G}$ is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of the perfect matching between the corresponding vertices of G and $\overline{G}$.

In this paper, we explore some topological indices under several graph operations for some connected graphs.

2. WIENER INDEX OF NEIGHBOURHOOD CORONA AND SPLITTING GRAPH

The corona of two graphs is defined in [5] and there have been some results on the corona of two graphs [7].

Theorem 2.1:

For any two graphs G_1 and G_2, the Wiener index of $G_1 * G_2$ is

$W(G_1 * G_2) = (n_2+1)^2 W(G_1) + n_1 W(G_2) + 2n_2^2 m_1 + 2n_1 n_2$

Proof:

By observing the neighbourhood corona operation $G_1 * G_2$ of any two graphs G_1 and G_2, we have

$d_{G_1}(v_i, v_j) = d_{G_1 * G_2}(u_{i,k}, v_j)$,

$$d_{G_1 * G_2}(u_{i,k}, u_{j,l}) = \begin{cases} d_{G_1}(v_i, v_j), & \text{if vertices are non adjacent} \\ d_{G_1}(v_i, v_j) + 2, & \text{if vertices are adjacent} \end{cases}$$ and

$d_{G_1}(v_i, v_j) = d_{G_1 * G_2}(v_i, v_j)$.

From these,

$$d_{G_1 * G_2}(v_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n_1} d_{G_1 * G_2}(v_i, v_j) + \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} d_{G_1 * G_2}(v_i, u_{j,k}) + \sum_{k=1}^{n_2} d_{G_1 * G_2}(v_i, u_{i,k})$$

$$= d_{G_1}(v_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} \sum_{k=1}^{n_1} d_{G_1}(v_i, v_j) + 2n_2$$

$$= d_{G_1}(v_i) + n_2 d_{G_1}(v_i) + 2n_2$$

$$= (n_2 + 1) d_{G_1}(v_i) + 2n_2 \qquad (1)$$ and

$$d_{G_1 * G_2}(u_{i,k}) = \sum_{\substack{l=1 \\ l \neq k}}^{n_2} d_{G_1 * G_2}(u_{i,k}, u_{i,l}) + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} \sum_{l=1}^{n_2} d_{G_1 * G_2}(u_{i,k}, u_{j,l}) + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} d_{G_1 * G_2}(u_{i,k}, v_j) + d_{G_1 * G_2}(v_i, u_{i,k})$$

$$= d_{G_2}(u_{i,k}) + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} d_{G_1}(v_i, v_j) + 2n_2 \deg_{G_1}(v_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} \sum_{l=1}^{n_2} d_{G_1}(v_i, v_j) + 2$$

$$= d_{G_2}(u_{i,k}) + n_2 \left[d_{G_1}(v_i) + 2\deg_{G_1}(v_i) \right] + d_{G_1}(v_i) + 2$$

$$= d_{G_2}(u_{i,k}) + (n_2 + 1) d_{G_1}(v_i) + 2n_2 \deg_{G_1}(v_i) + 2. \qquad (2)$$

From (1) and (2), the Wiener index of $G_1 * G_2$ is

$$W(G_1 * G_2) = \frac{1}{2}\left[\sum_{i=1}^{n_1} d_{G_1 * G_2}(v_i) + \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} d_{G_1 * G_2}(u_{i,k}) \right]$$

$$=\frac{1}{2}\left[\sum_{i=1}^{n_1}(n_2+1)d_{G_1}(v_i)+2n_2\right]+\frac{1}{2}\left[\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\left(d_{G_2}(u_{i,k})+(n_2+1)d_{G_1}(v_i)+2n_2\deg_{G_1}(v_i)+2\right)\right]$$

$$=\frac{1}{2}\left[2(n_2+1)\sum_{i=1}^{n_1}d_{G_1}(v_i)+2n_1n_2\right]+\frac{1}{2}\left[\sum_{i=1}^{n_1}\left(\sum_{k=1}^{n_2}d_{G_2}(u_{i,k})+n_2(n_2+1)d_{G_1}(v_i)+2n_2^2\deg_{G_1}(v_i)+2n_2\right)\right]$$

$$=\left[(n_2+1)W(G_1)+n_1n_2\right]+\frac{1}{2}\left[\sum_{i=1}^{n_1}\left(2W(G_2)+n_2(n_2+1)d_{G_1}(v_i)+2n_2^2\deg_{G_1}(v_i)+2n_2\right)\right]$$

$$=\left[(n_2+1)W(G_1)+n_1n_2\right]+\sum_{i=1}^{n_1}\left[W(G_2)+\frac{n_2(n_2+1)}{2}d_{G_1}(v_i)+n_2^2\deg_{G_1}(v_i)+n_2\right]$$

$$=\left[(n_2+1)W(G_1)+n_1n_2\right]+\left[n_1W(G_2)+n_2(n_2+1)W(G_1)+2n_2^2m_1+n_1n_2\right]$$

$$=(n_2+1)W(G_1)+n_1W(G_2)+n_2(n_2+1)W(G_1)+2n_2^2m_1+2n_1n_2$$

$$=(n_2+1)^2W(G_1)+n_1W(G_2)+2n_2^2m_1+2n_1n_2$$

□

Corollary 2.2:

The Wiener index for the splitting graph of a graph G is $W(S'(G))=4W(G)+2m+2n$.

Proof:

By taking $G_1=G, G_2=K_1$ and G_1*G_2 is the splitting graph of G_1, the result follows. □

Theorem 2.3:

For a graph *G*, the 3-Steiner Wiener index of the complementary prism $G\overline{G}$ is

$$W_3(G\overline{G})=2\left[W_3(G)+W_3(\overline{G})\right]+\frac{2}{3}\left[W(G)+W(\overline{G})\right]+2n\binom{n}{3}+2n(n-1).$$

Proof:

Let $d_G^S(v)$ denote the sum of the Steiner distances of the sets of cardinality 3 containing v in *G*. Then,

$$d_{G\overline{G}}^S(v)=d_G^S(v)+d_{\overline{G}}^S(\overline{v})+\binom{n}{3}+d_{\overline{G}}(\overline{v})+n-1 \text{ and}$$

$$d_{G\overline{G}}^S(\overline{v})=d_G^S(v)+d_{\overline{G}}^S(\overline{v})+\binom{n}{3}+d_G(v)+n-1.$$

Therefore, $W_3(G\overline{G})=\frac{1}{3}\sum_{v\in V(G\overline{G})}d_{G\overline{G}}^S(v)$

$$=\frac{1}{3}\left[\sum_{v\in V(G)} d^S_{G\overline{G}}(v)+\sum_{\overline{v}\in V(\overline{G})} d^S_{G\overline{G}}(\overline{v})\right]$$

$$=\frac{1}{3}\left[\sum_{v\in V(G)}\left(d^S_G(v)+d^S_{\overline{G}}(\overline{v})+\binom{n}{3}+d_{\overline{G}}(\overline{v})+n-1\right)+\sum_{\overline{v}\in V(\overline{G})}\left(d^S_{\overline{G}}(\overline{v})+d^S_G(v)+\binom{n}{3}+d_G(v)+n-1\right)\right]$$

$$=W_3(G)+W_3(\overline{G})+n\binom{n}{3}+\frac{2W(\overline{G})}{3}+n(n-1)+W_3(G)+W_3(\overline{G})+n\binom{n}{3}+\frac{2W(G)}{3}+n(n-1)$$

$$=2\left[W_3(G)+W_3(\overline{G})\right]+\frac{2}{3}\left[W(G)+W(\overline{G})\right]+2n\binom{n}{3}+2n(n-1).\qquad\square$$

Theorem 2.4:

Let G be a graph and xy be an edge of G so that $G-xy$ has two components namely G_1 and G_2. Then,

$$W_3(G)=W_3(G_1)+W_3(G_2)+\frac{2}{3}\left[W(G_1)+W(G_2)\right]$$
$$+\frac{1}{3}\left[(n_1+n_2)\left(\binom{n_1}{2}+\binom{n_2}{2}\right)+\left(n_1\binom{n_2}{2}+n_2\binom{n_1}{2}\right).\left(d_{G_1}(x)+d_{G_2}(y)\right)+n_1d^S_{G_2}(y)+n_2d^S_{G_1}(x)\right]$$

Proof:

Let $u_1,u_2,u_3,\ldots u_{n_1}$ and $v_1,v_2,v_3,\ldots v_{n_2}$ be the vertices of G_1 and G_2 in $G-xy$.

Then

$$d^S_G(u_i)=d^S_{G_1}(u_i)+\binom{n_2}{2}+\binom{n_2}{2}d_{G_1}(x)+d^S_{G_2}(y)+d_{G_1}(u_i)+\binom{n_1}{2}+\binom{n_1}{2}d_{G_2}(y)\text{ and}$$

$$d^S_G(v_i)=d^S_{G_2}(v_i)+\binom{n_1}{2}+\binom{n_1}{2}d_{G_2}(y)+d^S_{G_1}(x)+d_{G_2}(v_i)+\binom{n_2}{2}+\binom{n_2}{2}d_{G_1}(x).$$

Therefore,

$$W_3(G)=\frac{1}{3}\left[\sum_{i=1}^{n_1}d^S_G(u_i)+\sum_{j=1}^{n_2}d^S_G(v_i)\right]$$

$$=\frac{1}{3}\left[\sum_{i=1}^{n_1}d^S_{G_1}(u_i)+n_1\binom{n_2}{2}+n_1\binom{n_2}{2}d_{G_1}(x)+n_1d^S_{G_2}(y)+\sum_{i=1}^{n_1}d_{G_1}(u_i)+n_1\binom{n_1}{2}+n_1\binom{n_1}{2}d_{G_2}(y)\right]+$$
$$\frac{1}{3}\left[\sum_{j=1}^{n_2}d^S_{G_2}(v_j)+n_2\binom{n_1}{2}+n_2\binom{n_1}{2}d_{G_2}(y)+n_2d^S_{G_1}(x)+\sum_{j=1}^{n_2}d_{G_2}(v_j)+n_2\binom{n_2}{2}+n_2\binom{n_2}{2}d_{G_1}(x)\right]$$

$$=\frac{1}{3}\begin{bmatrix}3W_3(G_1)+n_1\left[\binom{n_1}{2}+\binom{n_2}{2}\right]+n_1\binom{n_2}{2}d_{G_1}(x)+n_1\binom{n_2}{2}d_{G_2}(y)+n_1d^S_{G_2}(y)+2W(G_1)+\\3W_3(G_2)+n_2\left[\binom{n_1}{2}+\binom{n_2}{2}\right]+n_2\binom{n_1}{2}d_{G_2}(y)+n_2\binom{n_1}{2}d_{G_1}(x)+n_2d^S_{G_1}(x)+2W(G_2)\end{bmatrix}.$$

$$=\frac{1}{3}\left[\begin{array}{l}3(W_3(G_1)+W_3(G_2))+(n_1+n_2)\left[\binom{n_1}{2}+\binom{n_2}{2}\right]+\left[n_1\binom{n_2}{2}+n_2\binom{n_1}{2}\right]d_{G_2}(y)+\\ \left[n_1\binom{n_2}{2}+n_2\binom{n_1}{2}\right]d_{G_1}(x)+n_1d^S_{G_2}(y)+n_2d^S_{G_1}(x)+2(W(G_1)+W(G_2))\end{array}\right]$$

$$=\frac{1}{3}\left[\begin{array}{l}3(W_3(G_1)+W_3(G_2))+(n_1+n_2)\left(\binom{n_1}{2}+\binom{n_2}{2}\right)+\left(n_1\binom{n_2}{2}+n_2\binom{n_1}{2}\right).(d_{G_1}(x)+d_{G_2}(y))+\\ n_1d^S_{G_2}(y)+n_2d^S_{G_1}(x)+2(W(G_1)+W(G_2))\end{array}\right]$$

Hence the result follows □

Theorem 2.5:

Let G be the graph obtained from G_1 by duplicating each vertex of G_1 by an edge.

Then $W_3(G)=9W_3(G_1)+2W(G)+\frac{8n^2}{3}+\frac{16n}{3}\binom{n}{3}+\frac{2}{3}n$

Proof:

Let, $v_1,v_2,\ldots,v_n$ be the vertices of G_1 and $\{v_i',v_i''\}$ be the duplicating edge corresponding to v_i in G_1. Then,

$$d^S_G(v_i)=d^S_{G_1}(v_i)+[d_{G_1}(v_i)+2n]+2d^S_{G_1}(v_i)+4\binom{n}{3}=3d^S_{G_1}(v_i)+2n+d_{G_1}(v_i)+4\binom{n}{3}$$ and

$$d^S_G(v_i')=d^S_G(v'')=d^S_{G_1}(v_i)+1+[d_{G_1}(v_i)+3n]+2d^S_{G_1}(v_i)+6\binom{n}{3}=3d^S_{G_1}(v_i)+3n+d_{G_1}(v_i)+6\binom{n}{3}+1$$

Therefore,

$$W_3(G)=\frac{1}{3}\sum_{v\in V G)}d^S_G(v)$$

$$=\frac{1}{3}\sum_{i=1}^{n}[d^S_G(v_i)+d^S_G(v_i')+d^S_G(v_i'')]$$

$$=\frac{1}{3}\left[3\sum_{i=1}^{n}d^S_{G_1}(v_i)+\sum_{i=1}^{n}d_{G_1}(v_i)+2n^2+4n\binom{n}{3}+6\sum_{i=1}^{n}d^S_{G_1}(v_i)+2\sum_{i=1}^{n}d_{G_1}(v_i)+6n^2+12n\binom{n}{3}+2n\right]$$

$$=\frac{1}{3}\left[27W_3(G_1)+6W(G)+8n^2+16n\binom{n}{3}+2n\right]$$

$$=9W_3(G_1)+2W(G)+\frac{8n^2}{3}+\frac{16n}{3}\binom{n}{3}+\frac{2n}{3}$$ □

3. FIRST ZAGREB INDEX OF NEIGHBOURHOOD CORONA

Theorem 3.1:

For any two graphs G_1 and G_2, the first Zagreb index of $G_1 * G_2$ is

$M_1(G_1 * G_2) = (n_2^2 + 3n_2 + 1)M_1(G_1) + n_1 M_1(G_2) + 8m_1m_2$.

Proof:

By observing the neighbourhood corona operation $G_1 * G_2$ of any two graphs G_1 and G_2, $\deg_{G_1*G_2}(v_i) = (n_2+1)\deg_{G_1}(v_i)$ and $\deg_{G_1*G_2}(u_{i,k}) = \deg_{G_1}(v_i) + \deg_{G_2}(u_{i,k})$.

Therefore, $M_1(G_1 * G_2) = \sum_{u \in V(G_1 * G_2)} \left[\deg_{G_1*G_2}(u)\right]^2$

$$= \sum_{i=1}^{n_1} \left[\deg_{G_1*G_2}(v_i)\right]^2 + \sum_{i=1}^{n_1}\sum_{k=1}^{n_2} \left[\deg_{G_1*G_2}(u_{i,k})\right]^2$$

$$= \sum_{i=1}^{n_1} \left[(n_2+1)^2(\deg_{G_1}(v_i))^2\right] + \sum_{i=1}^{n_1}\sum_{k=1}^{n_2} \left[\deg_{G_1}(v_i) + \deg_{G_2}(u_{i,k})\right]^2$$

$$= (n_2+1)^2 M_1(G_1) + \sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\left[(\deg_{G_1}(v_i))^2 + (\deg_{G_2}(u_{i,k}))^2 + 2\deg_{G_1}(v_i).\deg_{G_2}(u_{i,k})\right]$$

$$= (n_2+1)^2 M_1(G_1) + n_2 M_1(G_1) + n_1 M_1(G_2) + 2\sum_{i=1}^{n_1}\sum_{k=1}^{n_2}\left[\deg_{G_1}(v_i).\deg_{G_2}(u_{i,k})\right]$$

$$= (n_2+1)^2 M_1(G_1) + n_2 M_1(G_1) + n_1 M_1(G_2) + 2\sum_{i=1}^{n_1}\left[\deg_{G_1}(v_i)(2m_2)\right]$$

$$= (n_2^2 + 3n_2 + 1)M_1(G_1) + n_1 M_1(G_2) + 8m_1m_2$$ □

Corollary3.2:

The first Zagreb index for the splitting graph of a graph G is

$M_1(S'(G)) = 5M_1(G_1) + n_1 M_1(G_2)$

REFERENCES:

[1] F. Buckley and F. Harary, *Distance in Graphs*, AddisonWesley, Redwood, 1990.

[2] G. Chartrand, and L. Lesniak, *Graphs & Digraphs*, 2nd Edition. Monterey, CA (1986).

[3] A.A. Dobrynin, R. Entringer and I. Gutman, *Wiener index of trees: Theory and applications,* Acta Appl. Math. 66 (2001) 211-249.

[4] A.A. Dobrynin, I. Gutman, S. Klavzar and P. Zigert, *Wiener index of hexagonal systems.* Acta Appl. Math. 72 (2002) 247-294.

[5] R. Frucht and F. Harary, *On the corona two graphs*, Aequationes Math., 4 (1970), 322-325.

[6] I. Gutman and N. Trinajstic, *Graph theory and molecular orbitals, Total Π electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1972) 535-538.

[7] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, MA/Menlo Park, CA/London, 1969.

[8] X.L. Li and I. Gutman, *Mathematical Aspects of Randic-Type Molecular Structure Descriptors*, MCM, Kragujevac, 2006.

[9] U. Mary, P.S. Arockiaraj and A. James Albert. *Wiener Polynomial for Steiner distance of Corona and complement graphs*, International Journal of Computer Application, 6(2)(2012) 105-112.

[10] E. Sampathkumar and H.B. Walikar, *On the splitting graph of a graph*, Karnataka Univ. J. Sci. 35/36 (1980-1981), 13-16.

[11] H. Wiener, *Structural determination of paraffin boiling points*, *J. Am. Chem. Soc.* 69 (1947) 17-20.