GI/M/1 Queue with multiple working vacation

K. Julia Rose Mary
Department of Mathematics, Nirmala College for women,
Coimbatore, INDIA -641 018. rosevictor@hotmail.com

M.I. Afthab Begum
Department of Mathematics, Avinashilingam University for Women,
Coimbatore, INDIA -641 043. sriyaz_2k@yahoo.com

Abstract

In this paper we analyze a GI/M/1 queue with working vacation, in which the server works at different rate rather than completely stopping the service during vacation periods. Baba investigated a GI/M/1 queue with multiple working vacations .They have formulated the queueing system as an embedded two dimensional Markov chain by choosing the arrival epoch as embedded points. Using the algorithmic approach Neutus and others derived the steady state distributions for the number of customers in the system both at arrival and arbitrary epochs and for the sojourn time for an arbitrary customer. Thus the purpose of this paper is to analyze GI/M/1 queueing model under single server **working vacation** and derived the equilibrium distribution of system size at pre arrival epoch by solving the difference equations using operator technique. Further the expected queue length and various performance measures are obtained in a closed form. Finally special cases are also deduced.

Key words : GI/M/1 Queue, Working vacation.

Mathematics Subject Classification : **90B22.**

1. Introduction

Over the past two decades, queueing systems with vacations have been studied by many researchers and have been applied to many situations , namely in computer systems, communication networks , production managing and so forth [2]. Servi and Finn [5] first studied queueing system with working vacations , where the server works at a lower rat rather than completely stopping service during vacation. That is during a working vacation, customers will undergo service at a lower rate and depart system , whereas, customers in the classical vacation will impossibly depart the system. Therefore the working vacation models have more complicated modalities and the analysis of this kind of models is more difficult than classical vacation queue.

Servi and Finn [5] studied an M/M/1 queue with multiple working vacations and obtained the p.g.f of the number of customers in the system and mean waiting time of customers, and applied the results to perform analysis of gateway router in fiber communication networks. Liu et al [4] discussed stochastic decomposition structures of stationary indices, derived the distribution of additional queue length and additional delay and obtained expected regular busy period and expected busy cycle. Kim, Choi and Chae [3], and Takagi [7] generalized the work of Servi and Finn's model to an M/G/1 queue with multiple working vacations . Baba [1] investigated a GI / M/1 queue with multiple working vacations and derived the steady state distributions for the number of customers in the system both at arrival and arbitrary epochs and for the sojourn time for an arbitrary customer.

Tian et al [6] analyzed an M/M/1 queue with single working vacation using quasi birth and death process and matrix geometric solution method, they derived the distributions for the number of customers and the virtual time in the system in steady state. Furthermore, they obtained the expected busy period, expected busy cycle, and got the stochastic decomposition structures of stationary indices.

In this paper, we have derived the steady state probabilities using embedded Markov chain technique and derived the expected queue length in a closed form. Further various performance measures are also deduced.

Model Formulation:

We consider GI/ M/1 queue in which the server begins an exponentially distributed working vacation, whenever the system becomes empty. During the working vacation the arriving customers are served at a mean rate μ_v. When the vacation ends, if there are customers in the queue, then server changes his service rate μ_v to μ_0 and regular busy period starts. Otherwise the server begins another working vacation. That is the server follows multiple (or) repeated vacation policy. Thus the service time during vacation and regular busy period and vacation follow exponential distributions with parameters μ_v, μ_0 and η respectively and they are independent of each other.

Embedded Markov chain of queue length:

Let τ_n denote the arrival epoch of the nth customer with $\tau_0 = 0$. The inter arrival times $\{ \tau_n , n \geq 1 \}$ are independent and identically distributed with a general distribution function denoted by A(t) with a mean $1/\lambda$ and a Laplace Stieltjes transform (LST) denoted by $A^*(\theta)$.

By considering the above assumptions and by utilizing the Embedded Markov Chain technique, the steady state probabilities , expected queue length and various performance measures are derived.

Queue length at pre arrival epochs:

τ_n , n = 1,2,3,....... ($\tau_0 = 0$) are the arrival epochs and we examine the system at the pre arrival epochs $\tau_n - 0$.

Let Q (τ) denote the number of customers in the system at time τ and $Q_n = Q(\tau_n - 0)$

and $\tau_n = \begin{cases} 0 & \text{if the nth arrival occurs during working vacation} \\ 1 & \text{if the nth arrival occurs during service period} \end{cases}$

Since the working vacation times, the service time during a regular busy period and working vacation are all exponentially distributed the process $\{ (Q_n, \tau_n), n \geq 1 \}$ defines a semi Markov chain with state space $\{ (n,0), n = 0,1,2,3,\ldots\} \cup \{ (n,1), n = 1,2,3,\ldots\ldots\}$

(i) Thus the state (n,0) ,($n \geq 1$) represents that there are n customers in the system and the server is on working vacation, whose service rate is μ_v.

(ii) The state (n,1) ,($n \geq 1$) denotes that the server is serving at regular rate μ_0 with **n** customers in the system.

(iii) (0,0) implies that the system is empty and the server is idle during vacation.

In order to obtain the steady state equations we first define the following probabilities :

1. Let $b_k = \int_0^\infty e^{-\mu_v t} \frac{(\mu_0 t)^k}{k!} dA(t)$ $k \geq 0$. Then b_k gives the probability that k customers are served at regular rate μ_0 in an inter arrival time .

2. Similarly $c_k = \int_0^\infty e^{-\eta t} \frac{(\mu_v t)^k}{k!} e^{\mu_v t} dA(t)$ $k \geq 0$ implies the probability that the working vacation time is greater than the inter- arrival time and k service completions occur at rate μ_v during an inter arrival time . and

3. $d_k = \{ \int_0^\infty \sum_{j=0}^{k} \int_0^t (\eta e^{-\eta x} \frac{(\mu_v x)^j}{j!} e^{-\mu_v x} \frac{(\mu_0 (t-x))^{k-j}}{(k-j)!} e^{-\mu_0 (t-x)}) \, dx \; dA(t) \}$ gives the probability that the server who is in working vacation returns from vacation after a time x ($0 \leq x \leq t$) and k customers are served in an inter arrival time t, in such a way that j customers are served in time x at a rate μ_v and the remaining (k – j) customers are served in time (t-x) at rate μ_0.

Steady state equations:

By following the law of transition, the Markov chain $\{(Q_n, \tau_n) ; n \geq 1\}$ leads to the steady state equations satisfied by the limiting probabilities

$p_{nj} = \lim_{k\to\infty} pr(n_k = j_k, j_k = j)$, $n \geq 0$, $j = 0,1$

Thus by assuming the steady state exists the Chapman Kolmogrov equations satisfied by p_{nj} 's in the steady state are given by

$$p_{00} = \sum_{i=0}^{\infty} p_{i0}\left(1 - \sum_{k=0}^{i} c_k + d_k\right) + \sum_{i=1}^{\infty} p_{i1}\left(1 - \sum_{k=0}^{i} b_k\right) \quad (1)$$

$$p_{n0} = \sum_{j=0}^{\infty} p_{j+n-1}\, c_j \qquad n \geq 1 \quad (2)$$

$$p_{11} = \sum_{i=1}^{\infty} p_{i1}(b_i) + \sum_{i=0}^{\infty} p_{i0}\ (d_i) \quad (3)$$

$$p_{n1} = \sum_{j=0}^{\infty} p_{j+n-1\,1}\ b_j + \sum_{j=0}^{\infty} p_{j+n-1\,0}\ d_j \quad n \geq 2 \quad (4)$$

Steady state solution:

To solve the steady state equations, we define the forward shifting operator E on p_{nj} by $E(p_{n0}) = p_{n+1\,0}$. Thus equation (2) can be written as

$$(E - \sum_{j=0}^{\infty} c_j E^j) p_{n0} = 0 \text{ for } n \geq 0 \quad (5)$$

The characteristic equation of the homogeneous difference equation (5) is

$$\phi(z) = z - \sum_{j=0}^{\infty} c_j z^j = 0$$

Let $C(z) = \sum_{j=0}^{\infty} c_j z^j$ be the p.g.f of the probabilities c_j's then $c(z) = A^*(\eta + \mu_v(1-z))$.

Since C(z) is monotonically increasing and strictly convex , there exists a unique root $r_1 \in (0,1)$ of $\phi(z) = 0$ provided $\phi'(1) > 0$ (Gross and Haris [2])

Since $\phi'(1) = 1 + \mu_v A^*(\eta) > 0$,

the solution of the homogeneous difference equation is given by

$$p_{n0} = r_1^{\,n}\ p_{n0} \quad n \geq 0 \quad (6)$$

with $C(r_1) = r_1$ and $r_1 \in (0, 1)$

Similarly by defining the forward displacement operator E on p_{n1},

the equation (4) can be written as ,

$$(E - \sum_{j=0}^{\infty} b_j E^j) p_{n1} = \sum_{j=0}^{\infty} p_{n+j\,0}\, d_j \qquad n \geq 1 \quad (7)$$

$$= \sum_{j=0}^{\infty} d_j E^j (p_{n0})$$

Hence equation (7) is a non – homogeneous difference equation whose characteristic equation is given by $z = \sum_{j=0}^{\infty} b_j z^j = B(z) = A^*(\mu_0(1-z))$.

Following the arguments mentioned earlier if $B'(1) = \frac{\mu_0}{\lambda} > 1$ then there exists an unique root $r_0 \in (0,1)$ for the characteristic equation $\varphi(z) = z - B(z) = 0$

$$i.e\ B(r_0) = r_0 \text{ and } 0 < r_0 < 1 \quad (8)$$

Thus the solution of the non- homogeneous difference equation (7) becomes

$p_{n1} = A_1 r_0^{\,n} + \sum_{j=0}^{\infty} \frac{d_j r_1^{\,n+j} p_{00}}{r_1 - \sum_{j=0}^{\infty} b_j r^j}$ provided $r_1 \neq r_0$ and $\rho_0 = \frac{\lambda}{\mu_0} < 1$

Thus p_{n1} can be written as $p_{n1} = (A r_0^n + k(r_1)\, r_1^n) p_{00}$ $n \geq 1$.

Where $A_1 = A\, p_{00}$ and $k(r_1) = \sum_{j=0}^{\infty} \frac{d_j r_1^{n+j} p_{00}}{r_1 - \sum_{j=0}^{\infty} b_j r^j}$ which can also be written as

$$k(r_1) = \frac{D(r_1)}{r_1 - B(r_1)} \qquad (9)$$

where $D(r_1) = \sum_{j=0}^{\infty} d_j r_1^j$

Hence the probabilities p_{n0}'s and p_{n1}'s are given by

$$\left.\begin{aligned} p_{n0} &= r_1^n\, p_{n0} \quad n \geq 0 \quad \text{and} \\ p_{n1} &= (A r_0^n + k(r_1)\, r_1^n) p_{00} \quad n \geq 1 \end{aligned}\right\} \qquad (10)$$

Thus the probabilities are expressed in terms of p_{00} and A . The constant A can be evaluated by using (1) or (3).

Substituting for p_{i0} ' s and p_{i1} 's from equation (10) in equation (3) we get

$$(A r_0 + k(r_1)\, r_1) = \sum_{i=1}^{\infty} (A r_0^i + k(r_1)\, r_1^i) b_i + \sum_{i=0}^{\infty} r_1^i d_i$$

$$A(r_0 - B(r_0)) = k(r_1)(B(r_1) - r_1) + D(r_1) - b_0 (A + k(r_1)) \qquad (11)$$

Equation (8) and (9) imply that

$$B(r_0) = r_0 \quad \text{and} \quad k(r_1) = \frac{D(r_1)}{r_1 - B(r_1)}$$

The equation (11) can be simplified as

$$k(r_1)(B(r_1) - r_1) + D(r_1) - b_0 (A + k(r_1)) = 0$$

$$\frac{D(r_1)}{r_1 - B(r_1)} (B(r_1) - r_1) + D(r_1) - b_0 (A + k(r_1)) = 0$$

Thus $b_0 (A + k(r_1)) = 0$, since $b_0 \neq 0$

we get $(A + k(r_1)) = 0 \Rightarrow A = -k(r_1)$

It is verified that equation (1) is also satisfied by noting that

$$\sum_{i=0}^{\infty} p_{i0} \sum_{k=0}^{i} c_k = \sum_{i=0}^{\infty} c_i \sum_{k=i}^{\infty} p_{k0}$$

$$\sum_{i=1}^{\infty} p_{i1} \sum_{k=1}^{i} b_k = \sum_{i=1}^{\infty} b_i \sum_{k=i}^{\infty} p_{k1}$$

Thus $p_{n1} = k(r_1)(r_1^n - r_0^n)\, p_{00}$ $n \geq 1$ (12)

Further the value of p_{00} can be calculated by using the normalizing condition

$\sum_{n=0}^{\infty} p_{n\,o} \; + \sum_{n=1}^{\infty} p_{n\,1} \; = \; 1$

Substituting for $p_{n\,o}$ *and* $p_{n\,1}$ from (10) we find that

$\frac{1}{(1\ \ r_1)} \; + \mathrm{k}(r_1)\left(\frac{r_1}{1\ \ r_1} - \frac{r_0}{1\ \ r_0}\right) = p_{0\,0}^{-1}$

Where $p_{00} = \frac{1-r_1}{1+k(r_1)\left(\frac{r_1-r_0}{1-r_0}\right)}$ (13)

Hence , $p_{n0} = r_1^{\,n}\; p_{n0} \quad n \geq 0$ and

$P_{n1} = k(r_1)\,(r_1^n - r_0^n)\, p_{00} \quad n \geq 1$ where p_{00} is given by equation (13) .

By using equation (14) the total pgf is given by

$p(z) \; = \; \sum_{n=0}^{\infty} p_{n\,o}\, z^n \; + \sum_{n=1}^{\infty} p_{n\,1}\, z^n$

$p(z) \; = \; \sum_{n=0}^{\infty} r_1^n p_{0\,o}\, z^n \; + \sum_{n=1}^{\infty} (Ar_0^n \; + \; k(r_1)\, r_1^n\,) p_{00}\, z^n$

$\quad = \; p_{00}\, [(1-r_1 z)^{-1} + \sum_{n=1}^{\infty} -k(r_1) r_0^n z^n \; + k(r_1)\, r_1^n\, z^n]$

$\quad = p_{00}\left[(1-r_1 z)^{-1} - k(r_1)\left\{\frac{r_0 z}{1-r_0 z} + \frac{r_1 z}{1-r_1 z}\right\}\right]$

Mean queue length:

In this section we calculate the mean queue size of the model.

If L_q denotes the mean queue size for the model then it can be written as

$L_q = \sum_{n=1}^{\infty} \; n\, p_{n1} \; + \; \sum_{n=0}^{\infty} n\, p_{n\,0}$

By substituting the value of p_{n1} and p_{n0} from equation (10),

L_q is simplified as ,

$L_q = \left[\frac{r_1}{(1-r_1)^2}\left(1 + \frac{D(r_1)}{r_1 - B(r_1)} - \frac{D(r_1) r_0}{(r_1 - B(r_1))(1-r_0)^2}\right)\right] p_{0\,0}$

where $p_{00} = \frac{1-r_1}{1+k(r_1)\left(\frac{r_1-r_0}{1-r_0}\right)}$

Other performance measures:

If p_v and p_b denote the probability that the server is on vacation and busy respectively then

$p_v = \sum_{n=0}^{\infty} p_{n\,0} \; = \frac{p_{0\,0}}{(1-r_1)}$;

$p_b = \sum_{n=1}^{\infty} p_{n1} = \sum_{n=1}^{\infty} k(r_1)(r_1^n - r_0^n)p_{00}$

by substituting for $k(r_1)$ and p_{00} we get

$$p_b = \frac{D(r_1)}{r_1 - B(r_1)}\left(\frac{r_1 - r_0}{(1-r_1)(1-r_0)}\right)$$

Particular cases:

If the arrival follows Poisson distribution the results of GI/M/1/MWV model Coincide with the corresponding results of M/M/1/MWV model of Servi and Finn [5].

Let $a(t) = \frac{dA(t)}{dt} = \lambda e^{-\lambda t}$

by applying the definition of Laplace transform for b_k we get $b_k = \frac{\lambda \mu_0^k}{(\lambda+\mu_0)^k}$

since $B(z) = \sum_{j=0}^{\infty} b_j z^j$ (from equation (7))

which in turn gives $B(z) = \frac{\lambda}{\lambda+\mu_v(1-z)}$

But from equation (8) we have $B(r_0) = r_0$

Hence we get $\mu_0 r_0^2 - (\lambda + \mu_0) r_0 + \lambda = 0$ as in M/M/1 where $r_0 = \rho_0$

Similarly by applying the definition of Laplace transform to c_k we get

$c_k = \frac{\lambda \mu_v^k}{(\lambda+\mu_v)^{k+1}}$, since $C(z) = \sum_{j=0}^{\infty} c_j z^j$ we get $C(z) = \frac{\lambda}{\lambda+\eta+\mu_v(1-z)}$

As $C(r_1) = r_1$, we obtain $\mu_v r_1^2 - (\lambda + \mu_v + \eta) r_1 + \lambda = 0$.

Again proceeding in the same way we get

$$d_k = \frac{\lambda \eta \mu_0^k}{(\lambda+\mu_v+\eta)(\lambda+\mu_0)} \sum_{j=0}^{k} \left(\frac{\mu_v(\lambda+\mu_0)}{\mu_0(\lambda+\eta+\mu_0)}\right)^k$$

Then $D(z) = \frac{\lambda\eta}{\lambda+\mu_0(1-z) \quad \lambda+\eta+\mu_v(1-z)}$

by substituting for B(z) and D(z) in equation (9) we get

$$k(r_1) = \frac{D(r_1)}{r_1 - B(r_1)} = \frac{-\eta r_1}{\mu_0(1-r_1)(\rho_0 - r_1)}$$ as in M/M/1/MWV

Thus we find that, $p_{n0} = r_1^n \; p_{n0} \quad n \geq 0$ and

$p_{n1} = k(r_1)(r_1^n - r_0^n)p_{00} \quad n \geq 1$ of GI/M/1/MWV

coincides with that of M/M/1/MWV of Servi and Finn [5].

Conclusion

In this paper we have developed the analytical steady state results for the GI/M/1 under multiple vacations in a closed form by solving its difference equations. Moreover we find that the results of M/M/1/MWV model of Servi and Finn [5] is deduced as a particular case.

References

[1] Y.Baba, Analysis of GI/M/1 queue with multiple working vacations, Operations Research Letters, 33 (2008) 201 – 209.

[2] Donald Gross Carl M. Harris, Fundamentals of Queueing Theory, Third Edition: John Wiley & Sons (Asia) P. Ltd.

[3] Kim .J. Choi, K.Chae, Analysis of queue length distributions of the M/G/1 queue with working vacations, International Conference on Statistics and related fields, Hawaii 2003.

[4] Ji Hong Li, Nai-shuotion, Zhe Geoge zhang, Hsing paul luh Analysis of the M/G/1 queue with exponentially working vacations – a matrix analytic approach, Queueing Systems, (2009) 139 -166.

[5] D.Servi, S.G.Finn, M/M/1 queue with working vacations (M/M/1/wv), Performance Evaluations, 50 (2002) 41 -52.

[6] N.Tian, X.Zhao, K.Wang, The M/M1 queue with single working vacation, Int. J. of Information and Management Science, 19(4) (2008) 621-634.

[7] H.Takagi, Queueing Analysis, A Foundation of Performance evaluation, vol.1 :Vacation and Priority Systems, Part I, Elsevier Science Publishers, Amsterdam 1991.