Optimization of EOQ Inventory Models with Two Backorders

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Abstract

The paper considers an inventory model with backorders in a fuzzy situation by employing the type of fuzzy numbers which are trapezoidal. The EOQ models have been developed using different optimization methods. A full fuzzy model is developed where the input parameters and the decision variables are fuzzified. The optimal policy for the developed model is determined using the Lagrangean conditions after the defuzzification of the cost function with the graded mean integration method. The proposed method finds the optimal lotsize and backorders level considering both linear and fixed backorders costs. Numerical examples are provided to highlight the difference between crisp and the fuzzy cases.

Keywords : Economic order quantity, backorders, fuzzy inventory function principle, graded mean integration representation, the total inventory cost.

Mathematics Subject Classification : **78M50, 90B05**

1. Introduction

The Economic Order Quantity (EOQ) developed by Harris [10]. EOQ based inventory models minimize the sum of mainly two costs, which are the holding and the ordering costs, L.F. Cardenas – Barron [2], S. Chand, J. Ward [3]. Park [13] proposed an EOQ model with the ordering and inventory holding cost being trapezoidal fuzzy numbers. Chang et al [6] considered an EOQ model with triangular fuzzy backorder. The works presented in this section mainly covers that optimizing the total cost function in an inventory system.

In this paper to derive the optimal lotsize, A.Banerjee [1], R.W. Crubbstro, A. Erdem [7] and the backorders level for the EOQ models considering two backorders costs, linear and fixed G.P. Sphicas [15]. Here demand and unit product cost, ordering cost, holding cost, backorder cost are represented as a trapezoidal fuzzy number. Chen [4] function principle is proposed for arithmetic operation of fuzzy number and Lagrangean method is used for optimization. Graded mean integration S.H. Chen, C.H. Hsieh [5] is used for defuzzifying the total inventory cost for the EOQ with backorders.

Section 1, deals with basic concepts of fuzzy sets, fuzzy numbers and function principle. Section 2 contains Graded Mean Representation Method. Section 3 proposes Optimal Order Quantity (Q) and Optimal Backorder Level (B). Section 4 discuss with, fuzzy EOQ inventory models with different situation. Section 5 contains a numerical example illustrates the solution procedure demonstrating that the developed model. Finally the conclusions are given in Section 6.

1.1. The Fuzzy Arithmetical Operations Under Function Principle

Function principle is proposed to be as the fuzzy arithmetical operations by trapezoidal fuzzy numbers. We define some fuzzy arithmetical operations under Function Principle as follows:

Suppose $\tilde{A} = (a_1, a_2, a_3, a_4)$ & $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers. Then

A.1 The addition of $\tilde{A}$ and $\tilde{B}$ is

$\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are any real numbers.

A.2 The multiplication of $\tilde{A}$ and $\tilde{B}$ is

$\tilde{A} \otimes \tilde{B} = (C_1, C_2, C_3, C_4)$

where $T = \{a_1b_1, a_2b_4, a_4b_1, a_4b_4\}$

$T_1 = \{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$

$C_1 = \min T, C_2 = \min T_1, C_3 = \max T_1, C_4 = \max T$

If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are all zero positive real numbers then

$\tilde{A} \otimes \tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$

A.3 $-\tilde{B} = (-b_4, -b_3, -b_2, -b_1)$ then the subtraction of $\tilde{A}$ and $\tilde{B}$ is

$\tilde{A} \ominus \tilde{B} = \{a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1\}$ where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are any real numbers.

A.4 $\frac{1}{\tilde{B}} = \tilde{B}^{-1} = \left(\frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\right)$ where b_1, b_2, b_3, b_4 are all positive real numbers. If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are all nonzero positive real numbers then the division $\tilde{A}$ and $\tilde{B}$ is

$$\tilde{A} \oslash \tilde{B} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right)$$

A.5 Let $\alpha \in R$, then

(i) $\alpha \geq 0, 0, \alpha \otimes \tilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4)$

(ii) $\alpha < 0, \alpha \otimes \tilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1)$

1.2. Extension of the Lagrangean Method

Taha [16] discussed how to solve the optimum solution of nonlinear programming problem with equality constraints by using Lagrangean Method, and showed how the Lagrangean method may be extended to solve inequality constraints. The general idea of extending the Lagrangean procedure is that if the unconstrained optimum the problem does not satisfy all constraints, the constrained optimum must occur at a boundary point of the solution space. Suppose that the problem is given by

Minimize $y = f(x)$

Sub to $g_i(x) \geq 0$, $i = 1, 2, \ldots, m$.

The nonnegativity constraints $x \geq 0$ if any are included in the m constraints. Then the procedure of the Extension of the Lagrangean method involves the following steps.

Step 1 : Solve the unconstrained problem

Min $y = f(x)$

If the resulting optimum satisfies all the constraints, stop because all constraints are redundant. Otherwise set $K = 1$ and go to step 2.

Step 2 : Activate any K constraints ((ie) convert them into equality) and optimize f(x) subject to the K active constraints by the Lagrangean method. If the resulting solution is feasible with respect to the remaining constraints and repeat the step. If all sets of active constraints taken K at a time are considered without encountering a feasible solution, go to step 3.

Step 3 : If $K = m$, stop ;

no feasible solution exists.

Otherwise set $K = K + 1$ and go to step 2.

2. Methodology

Graded Mean Integration Representation Method

Chen & Hsieh [5] introduced Graded mean Integration Representation Method based on the integral value of graded mean h-level of generalized fuzzy number for defuzzifying generalized fuzzy number. Here, we first define generalized fuzzy number as follows:

Suppose $\tilde{A}$ is a generalized fuzzy number as shown in Fig.1. It is described as any fuzzy subset of the real line R, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions.

i. $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to [0, 1],

ii. $\mu_{\tilde{A}}(x) = 0, -\infty < x \leq a_1$,

iii. $\mu_{\tilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$,

iv. $\mu_{\tilde{A}}(x) = W_A, a_2 \leq x \leq a_3$,

v. $\mu_{\tilde{A}}(x) = R(x)$ is strictly decreasing on $[a_3, a_4]$,

vi. $\mu_{\tilde{A}}(x) = 0, a_4 \leq x < \infty$,

where $0 < W_A \leq 1$ and a_1, a_2, a_3 and a_4 are real numbers.

This type of generalized fuzzy numbers are denoted as e $\tilde{A} = (a_1, a_2, a_3, a_4 ; \omega_A)_{LR}$ and $\tilde{A} = (a_1, a_2, a_3, a_4 : w_A)_{LR}$. When $\omega_A = 1$, it can be formed as $\tilde{A} = (a_1, a_2, a_3, a_4 ; \omega_A)_{LR}$. Second, by Graded Mean Integration Representation Method, L^{-1} and R^{-1} are the inverse functions of L and R respectively and the graded mean h-level value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4 ; \omega_A)_{LR}$ is given by $\frac{h}{2}\left(L^{-1}(h) + R^{-1}(h)\right)$. (see fig.1).

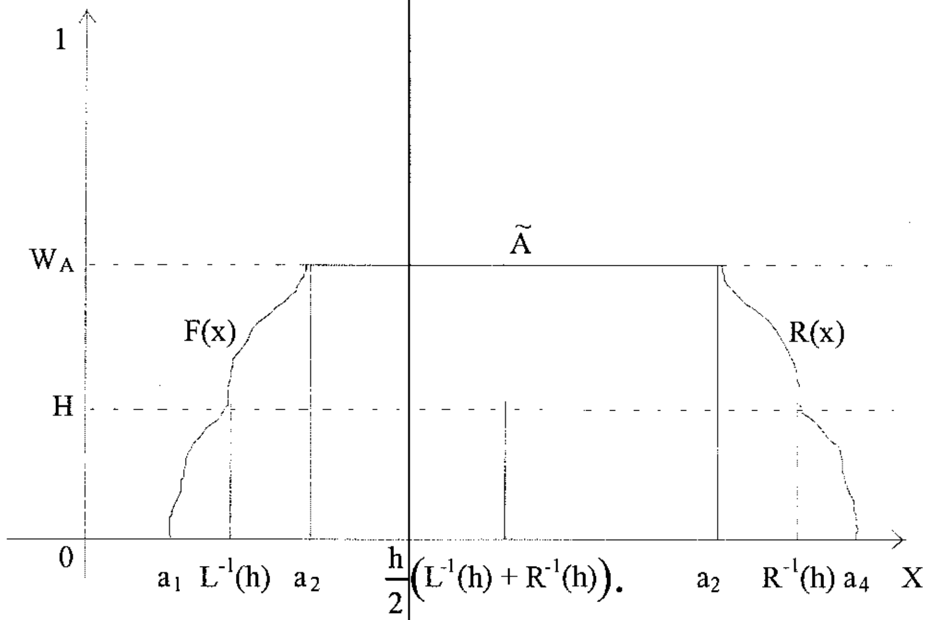

Fig.1. The graded mean h-level value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4 : w_A)_{LR}$.

Then the graded Mean Integration Representation of $P(\tilde{A})$ with grade w_A, where

$$P(\tilde{A}) = \frac{\int_0^{\omega_A} \frac{h}{2}\left(L^{-1}(h) + R^{-1}(h)\right) dh}{\int_0^{\omega_A} h\, dh} \qquad \ldots (2.1)$$

where $0 < h \le w_A$ and $0 < w_A \le 1$.

Throughout this paper, we only use popular trapezoidal fuzzy number as the type of all fuzzy parameters in our proposed fuzzy production inventorymodels. Let $\tilde{B}$ be a trapezoidal fuzzy number and be denoted as $\tilde{B} = (b_1, b_2, b_3, b_4)$. Then we can get the Graded Mean Integration Representation of $\tilde{B}$ by the formula (1) as

$$P(\tilde{B}) = \frac{\int_0^{1} \frac{h}{2}\left[(b_1+b_4) + h(b_2 - b_1 - b_4 + b_3)\right] dh}{\int_0^{1} h\, dh} = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6} \qquad \ldots (2.2)$$

3. The EOQ Inventory Model with Backorders

In this section, we develop a sequential optimization method using Lagrangean method. It consists of two stages. In the first stage the backorders level (B) is optimized using a concept of Lagrangean method. In the second stage we determine the optimal lotsize (Q).

Notations : The following notations are used throughout to develop the EOQ Leopoldo Edwardo Cardenas – Barron [12], Y.F. Huang [11] inventory model

d → demand rate per time unit

c → per unit product cost

A → ordering cost per order

h → per unit holding cost per time unit

$\hat{\pi}$ → per unit backorder cost per time unit (linear)

π → per unit backorder cost (fixed)

Q → order quantity

B → backorder level

Mathematical Model

The total inventory cost function for the EOQ with backorders give by

$$TC(Q, B) = \frac{Ad}{Q} + \frac{h(Q - B)^2}{2Q} + \frac{\hat{\pi} B^2}{2Q} + \frac{\pi Bd}{Q} + cd \quad \text{. . . (3.1)}$$

Equivalently, the total inventory cost can also be written as

$$TC(Q, B) = \frac{Ad}{Q} + \frac{hQ}{2} + B^2\left[\frac{h + \hat{\pi}}{2Q}\right] + B\left[\frac{\pi d - hQ}{Q}\right] + cd \quad \text{. . . (3.2)}$$

The objective is to find the optimal order quantity and optimal backorder level which maximize the total inventory cost function. The necessary conditions for minimum

$$\frac{\partial TC(Q, B)}{\partial B} = 0 \text{ and } \frac{\partial TC(Q, B)}{\partial B} = 0$$

Therefore the optimal backorder level is

$$B^* = \frac{hQ - \pi d}{h + \hat{\pi}} \quad \text{. . . (3.3)}$$

and the optimal order quantity is

$$Q^* = \sqrt{\frac{2Ad + B^2(h + \hat{\pi}) + 2BAd}{h}} \quad \text{. . . (3.4)}$$

Substituting (3) in (4)

$$Q^* = \sqrt{\frac{2Ad(h + \hat{\pi}) - (\pi d)^2}{h\,\hat{\pi}}}$$

Throughout this paper, we use the following variables in order to simplify the treatment of the fuzzy inventory models $\tilde{d}, \tilde{c}, \tilde{A}, \tilde{h}, \tilde{\hat{\pi}}, \tilde{\pi}$ are fuzzy parameters. The fuzzy total inventory cost function for the EOQ with backorders.

$$\widetilde{TC}(Q, B) = \left\{\frac{A_1d_1}{Q}+\frac{h_1Q}{2}+\frac{B^2(h_1+\hat{\pi}_1)}{2Q}+\frac{B\pi_1d_1}{Q} - Bh_4 + C_1d_1,\right.$$

$$\frac{A_2d_2}{Q}+\frac{h_2Q}{2}+\frac{B^2(h_2+\hat{\pi}_2)}{2Q}+\frac{B\pi_2d_2}{Q} - Bh_3 + C_2d_2,$$

$$\frac{A_3d_3}{Q}+\frac{h_3Q}{2}+\frac{B^2(h_3+\hat{\pi}_3)}{2Q}+\frac{B\pi_3d_3}{Q} - Bh_2 + C_3d_3,$$

$$\left.\frac{A_4d_4}{Q}+\frac{h_4Q}{2}+\frac{B^2(h_4+\hat{\pi}_4)}{2Q}+\frac{B\pi_4d_4}{Q} - Bh_1 + C_4d_4\right\}$$

$$\widetilde{TC}(Q, B) = [(A \otimes d) \oslash Q] \oplus \left[\frac{h}{2} \otimes Q\right] \oplus \left\{\frac{B^2}{2} \otimes (h_2 + \hat{\pi}) \% Q\right\}$$

$$\oplus \left[(B \otimes \pi \otimes d) \% Q\right] \; ! \; \left[(B \otimes h)\right] \oplus \left[(C \otimes d)\right] \qquad \ldots (3.5)$$

where $\oslash, \otimes, \oplus, \ominus$ are the fuzzy arithmetical operations under function principle,

Suppose $\tilde{A} = (A_1, A_2, A_3, A_4)$; $\tilde{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$

$\tilde{d} = (d_1, d_2, d_3, d_4)$; $\tilde{\hat{\pi}} = (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)$

$\tilde{h} = (h_1, h_2, h_3, h_4)$; $\tilde{c} = (c_1, c_2, c_3, c_4)$

are nonnegative trapezoidal fuzzy numbers. Then we solve the optimal order quantity and optimal backorder level of formula (7) as the following steps. Second, we defuzzify the fuzzy total inventory cost and backorders level by (4).

Graded mean integration representation of $\widetilde{TC}(Q, B)$ is

$$P(\widetilde{TC}(Q, B)) = \frac{1}{6}\left\{\left[\frac{A_1d_1}{Q}+\frac{h_1Q}{2}+\frac{B^2(h_1+\hat{\pi}_1)}{2Q}+\frac{B\pi_1d_1}{Q} - Bh_4 + C_1d_1\right]\right.$$

$$+2\left[\frac{A_2d_2}{Q}+\frac{h_2Q}{2}+\frac{B^2(h_2+\hat{\pi}_2)}{2Q}+\frac{B\pi_2d_2}{Q} - Bh_3 + C_2d_2\right]$$

$$+2\left[\frac{A_3d_3}{Q}+\frac{h_3Q}{2}+\frac{B^2(h_3+\hat{\pi}_3)}{2Q}+\frac{B\pi_3d_3}{Q} - Bh_2 + C_3d_3\right]$$

$$+\left[\frac{A_4d_4}{Q}+\frac{h_4Q}{2}+\frac{B^2(h_4+\hat{\pi}_4)}{2Q}+\frac{B\pi_4d_4}{Q} - Bh_1 + C_4d_4\right]\Bigg\}$$

Third, we can get the optimal order quantity Q^* and optimal backorder level B^* when $P(\widetilde{TC}(Q, B))$ is minimization. In order to find the minimization of $P(\widetilde{TC}(Q, B))$ the derivative of $P(\widetilde{TC}(Q, B))$ with Q and B is

$$\frac{\partial P\left(\widetilde{TC}(Q, B)\right)}{\partial B} = 0 \text{ and } \frac{\partial P\left(\widetilde{TC}(Q, B)\right)}{\partial Q} = 0$$

We find the optimal order quantity Q and optimal backorder level B.

$$Q^* = \sqrt{\frac{2(A_1d_1+2A_2d_2+2A_3d_3+A_4d_4)\left[(h_1+\hat{\pi}_1)+2(h_2+\hat{\pi}_2)+2(h_3+\hat{\pi}_3)+(h_4+\hat{\pi}_4)\right] - \left(\pi_1d_1+2\pi_2d_2+2\pi_3d_3+\pi_4d_4\right)^2}{(h_1+2h_2+2h_3+h_4)(\hat{\pi}_1+2\hat{\pi}_2+2\hat{\pi}_3+\hat{\pi}_4)}} \quad \ldots (3.6)$$

$$B^* = \frac{Q(h_1+2h_2+2h_3+h_4)-\left(\pi_1d_1+2\pi_2d_2+2\pi_3d_3+\pi_4d_4\right)}{(h_1+\hat{\pi}_1)+2(h_2+\hat{\pi}_2)+2(h_3+\hat{\pi}_3)+(h_4+\hat{\pi}_4)} \quad \ldots (3.7)$$

4. Fuzzy Inventory EOQ Model with Crisp Order Quantity and Fuzzy Backorder Quantity

In this section, we introduce the fuzzy inventory EOQ models R.J. Ronald, G.K. Yang, P. Chu [14] by changing the crisp backorder quantity into fuzzy backorder quantity. Suppose fuzzy backorder quantity $\widetilde{B}$ be the trapezoidal fuzzy number, $\widetilde{B} = (B_1, B_2, B_3, B_4)$ with $0 < B_1 \le B_2 \le B_3 \le B_4$. Then we get the fuzzy total inventory cost function for the EOQ,

$$P(\widetilde{TC}(Q, \widetilde{B})) = \Bigg\{\left(\frac{A_1d_1}{Q}+\frac{h_1Q}{2}+\frac{B_1^2(h_1+\hat{\pi}_1)}{2Q}+\frac{B_1\pi_1d_1}{Q} - B_4h_4 + C_1d_1\right),$$

$$\left(\frac{A_2d_2}{Q}+\frac{h_2Q}{2}+\frac{B_2^2(h_2+\hat{\pi}_2)}{2Q}+\frac{B_2\pi_2d_2}{Q} - B_3h_3 + C_2d_2\right),$$

$$\left(\frac{A_3d_3}{Q}+\frac{h_3Q}{2}+\frac{B_3^2(h_3+\hat{\pi}_3)}{2Q}+\frac{B_3\pi_3d_3}{Q} - B_2h_2 + C_3d_3\right),$$

$$\left(\frac{A_4d_4}{Q}+\frac{h_4Q}{2}+\frac{B_4^2(h_4+\hat{\pi}_4)}{2Q}+\frac{B_4\pi_4d_4}{Q} - B_1h_1 + C_4d_4\right)\Bigg\}$$

We can apply the Graded mean integration representation if $P(\widetilde{TC}(Q, \tilde{B}))$ by formula

(2) with $0 < B_1 \le B_2 \le B_3 \le B_4$. It will not change the meaning of formula if we replace inequality conditions $0 < B_1 \le B_2 \le B_3 \le B_4$ into the following inequality. $B_2 - B_1 \ge 0$, $B_3 - B_2 \ge 0$, $B_4 - B_3 \ge 0$, $B_1 > 0$. . In the following steps, extension of the Lagrangean method is used to find the solutions of B_1, B_2, B_3 and B_4 to minimize $P(\widetilde{TC}(Q, \tilde{B}))$.

Step 1 : Solve the unconstraint problem.

To find the min $[P(\widetilde{TC}(Q, \tilde{B}))]$, we have to find the derivative of $P(\widetilde{TC}(Q, \tilde{B}))$ with respect to B_1, B_2, B_3, B_4.

$$\frac{\partial P}{\partial B_1} = \frac{1}{6}\left[\frac{\partial B_1 (h_1 + \hat{\pi}_1)}{2Q} + \frac{\pi_1 d_1}{Q} - h_1\right]$$

$$\frac{\partial P}{\partial B_2} = \frac{2}{6}\left[\frac{\partial B_2 (h_2 + \hat{\pi}_2)}{2Q} + \frac{\pi_2 d_2}{Q} - h_2\right]$$

$$\frac{\partial P}{\partial B_3} = \frac{2}{6}\left[\frac{\partial B_3 (h_3 + \hat{\pi}_3)}{2Q} + \frac{\pi_3 d_3}{Q} - h_3\right]$$

$$\frac{\partial P}{\partial B_4} = \frac{1}{6}\left[\frac{\partial B_4 (h_4 + \hat{\pi}_4)}{2Q} + \frac{\pi_4 d_4}{Q} - h_4\right]$$

Let all the above partial derivatives equal to zero and solve B_1, B_2, B_3, B_4.

Therefore $$B_1 = \frac{Qh_1 - \pi_1 d_1}{h_1 + \hat{\pi}_1} ; \qquad B_2 = \frac{2Qh_2 - 2\pi_2 d_2}{2(h_2 + \hat{\pi}_2)}$$

$$B_3 = \frac{2Qh_3 - 2\pi_3 d_3}{2(h_3 + \hat{\pi}_3)} ; \qquad B_4 = \frac{Qh_4 - \pi_4 d_4}{h_4 + \hat{\pi}_4}$$

Because the above show that $B_1 > B_2 > B_3 > B_4$. It does not satisfy the constraint $0 < B_1 \le B_2 \le B_3 \le B_4$. Therefore set K = 1 and go to step 2.

Step 2 : Convert the inequality constraint $B_2 - B_1 \ge 0$ into equality constraint $B_2 - B_1 = 0$ and optimize $P(\widetilde{TC}(Q, \tilde{B}))$. Subject to $B_2 - B_1 = 0$ by the Lagrangean Method.

$$L(B_1, B_2, B_3, B_4, \lambda) = P(\widetilde{TC}(Q, \tilde{B})) - \lambda(B_2 - B_1)$$

Taking the partial derivatives of $L(B_1, B_2, B_3, B_4, \lambda)$ with respect to B_1, B_2, B_3, B_4 and λ to find the minimization of $L(B_1, B_2, B_3, B_4, \lambda)$.

Let all the above partial derivatives $\frac{\partial L}{\partial B_1}, \frac{\partial L}{\partial B_2}, \frac{\partial L}{\partial B_3}, \frac{\partial L}{\partial B_4}, \frac{\partial L}{\partial \lambda}$ equal to zero and solve to B_1, B_2, B_3, B_4. Then we get

$$B_1 = B_2 = \frac{(Qh_1 + 2Qh_2) - (\pi_1 d_1 + 2\pi_2 d_2)}{(h_1 + \hat{\pi}_1) + 2(h_2 + \hat{\pi}_2)};$$

$$B_3 = \frac{2Qh_3 - 2\pi_3 d_3}{2(h_3 + \hat{\pi}_3)}; \qquad B_4 = \frac{Qh_4 - \pi_4 d_4}{h_4 + \hat{\pi}_4}$$

Because the above show that $B_3 > B_4$ it does not satisfy the constraint $0 < B_1 \le B_2 \le B_3 \le B_4$. Therefore it is not a local optimum, set $K = 2$ and go to step 3.

<u>**Step 3 :**</u> Convert the inequality constraint $B_2 - B_1 \ge 0$, $B_3 - B_2 \ge 0$ into equality constraints $B_2 - B_1 = 0$ and $B_3 - B_1 = 0$. We optimize $P(\widetilde{TC}(Q, \tilde{B}))$ subject to $B_2 - B_1 = 0$ and $B_3 - B_2 = 0$ by the Lagrangean Method. Then the Lagrangean Method is

$$L(B_1, B_2, B_3, B_4, \lambda_1, \lambda_2) = P(\widetilde{TC}(Q, \tilde{B})) - \lambda_1 (B_2 - B_1) - \lambda_2 (B_3 - B_2).$$

In order to find the minimization of $L(B_1, B_2, B_3, B_4, \lambda_1, \lambda_2)$. We take the partial derivatives of $L(B_1, B_2, B_3, B_4, \lambda_1, \lambda_2)$ with respect to $B_1, B_2, B_3, B_4, \lambda_1, \lambda_2$ and let all the partial derivatives

$\frac{\partial L}{\partial B_1}, \frac{\partial L}{\partial B_2}, \frac{\partial L}{\partial B_3}, \frac{\partial L}{\partial B_4}, \frac{\partial L}{\partial \lambda_1}, \frac{\partial L}{\partial \lambda_2}$ equal to zero and to solve B_1, B_2, B_3, B_4.

$$B_1 = B_2 = B_3 = \frac{(Qh_1 + 2Qh_2 + 2Qh_3) - (\pi_1 d_1 + 2\pi_2 d_2 + 2\pi_3 d_3)}{(h_1 + \hat{\pi}_1) + 2(h_2 + \hat{\pi}_2) + 2(h_3 + \hat{\pi}_3)};$$

$$B_4 = \frac{Qh_4 - \pi_4 d_4}{h_4 + \hat{\pi}_4}$$

The above result $B_1 > B_4$ does not satisfy the constraint $0 < B_1 \le B_2 \le B_3 \le B_4$. Therefore set $K = 3$ and go to step 4.

<u>**Step 4 :**</u> Convert the inequality constraint $B_2 - B_1 \ge 0$, $B_3 - B_2 \ge 0$ and $B_4 - B_3 \ge 0$ into equality constraints $B_2 - B_1 = 0$, $B_3 - B_2 = 0$, $B_4 - B_3 = 0$. We optimize $P(\widetilde{TC}(Q, \tilde{B}))$ subject to $B_2 - B_1 = 0$, $B_3 - B_2 = 0$, $B_4 - B_3 = 0$ by the Lagrangean Method. The Lagrangean Function is given by $L(B_1, B_2, B_3, B_4, \lambda_1, \lambda_2, \lambda_3) = P(\widetilde{TC}(Q, \tilde{B})) - \lambda_1 (B_2 - B_1) - \lambda_2 (B_3 - B_2) - \lambda_3 (B_4 - B_3)$

In order to find the minimization of $L(B_1, B_2, B_3, B_4, \lambda_1, \lambda_2, \lambda_3)$. We take the partial derivatives of $L(B_1, B_2, B_3, B_4, \lambda_1, \lambda_2, \lambda_3)$ with respect to $B_1, B_2, B_3, B_4, \lambda_1, \lambda_2$ and λ_3. Let all the partial derivatives $\frac{\partial L}{\partial B_1}, \frac{\partial L}{\partial B_2}, \frac{\partial L}{\partial B_3}, \frac{\partial L}{\partial B_4}, \frac{\partial L}{\partial \lambda_1}, \frac{\partial L}{\partial \lambda_2}, \frac{\partial L}{\partial \lambda_3}$ equal to zero.

$$B^* = B_1 = B_2 = B_3 = B_4 = \frac{(h_1 + 2h_2 + 2h_3 + h_4)Q^* - (\pi_1 d_1 + 2\pi_2 d_2 + 2\pi_3 d_3 + \pi_4 d_4)}{(h_1 + \hat{\pi}_1) + 2(h_2 + \hat{\pi}_2) + 2(h_3 + \hat{\pi}_3) + (h_4 + \hat{\pi}_4)}.$$

4.1. Fuzzy Inventory EOQ Model With Crisp Backorder Quantity And Fuzzy Order Quantity

In this section, we introduce the fuzzy inventory EOQ models by changing the crisp order quantity into fuzzy order quantity suppose fuzzy order quantity $\tilde{Q}$ be a trapezoidal fuzzy number $\tilde{Q} = (Q_1, Q_2, Q_3, Q_4)$ with $0 < Q_1 \le Q_2 \le Q_3 \le Q_4$. The we can get the fuzzy total inventory cost function for the EOQ

$$P(\tilde{TC}(\tilde{Q}, B)) = \left\{\left(\frac{A_1 d_1}{Q_4} + \frac{h_1 Q_1}{2} + \frac{B^2(h_1 + \hat{\pi}_1)}{2Q_4} + \frac{B\pi_1 d_1}{Q_4} - Bh_4 + C_1 d_1\right),\right.$$
$$\left(\frac{A_2 d_2}{Q_3} + \frac{h_2 Q_2}{2} + \frac{B^2(h_2 + \hat{\pi}_2)}{2Q_3} + \frac{B\pi_2 d_2}{Q_3} - Bh_3 + C_2 d_2\right),$$
$$\left(\frac{A_3 d_3}{Q_2} + \frac{h_3 Q_3}{2} + \frac{B^2(h_3 + \hat{\pi}_3)}{2Q_2} + \frac{B\pi_3 d_3}{Q_2} - Bh_2 + C_3 d_3\right),$$
$$\left.\left(\frac{A_4 d_4}{Q_1} + \frac{h_4 Q_4}{2} + \frac{B^2(h_4 + \hat{\pi}_4)}{2Q_1} + \frac{B\pi_4 d_4}{Q_1} - Bh_1 + C_4 d_4\right)\right\}$$

We can obtain the Graded Mean Integration Representation of $P(\tilde{TC}(\tilde{Q}, B))$ by formula (2) with $0 < Q_1 \le Q_2 \le Q_3 \le Q_4$. In the following steps, extension of the Lagrangean Method is used to find the solutions of Q_1, Q_2, Q_3 and Q_4 to minimize $P(\tilde{TC}(\tilde{Q}, B))$.

Step 1 :

Solve the unconstraint problem

$$\min [P(\tilde{TC}(\tilde{Q}, B))] = \frac{1}{6}\left\{\frac{A_1 d_1}{Q_4} + \frac{h_1 Q_1}{2} + \frac{B^2(h_1 + \hat{\pi}_1)}{2Q_4} + \frac{B\pi_1 d_1}{Q_4} - Bh_4 + C_1 d_1\right.$$

$$+2\left(\frac{A_2d_2}{Q_3}+\frac{h_2Q_2}{2}+\frac{B^2(h_2+\hat{\pi}_2)}{2Q_3}+\frac{B\pi_2d_2}{Q_3}-Bh_3+C_2d_2\right)$$

$$+2\left(\frac{A_3d_3}{Q_2}+\frac{h_3Q_3}{2}+\frac{B^2(h_3+\hat{\pi}_3)}{2Q_2}+\frac{B\pi_3d_3}{Q_2}-Bh_2+C_3d_3\right)$$

$$+\frac{A_4d_4}{Q_1}+\frac{h_4Q_4}{2}+\frac{B^2(h_4+\hat{\pi}_4)}{2Q_1}+\frac{B\pi_4d_4}{Q_1}-Bh_1+C_4d_4\Bigg\}$$

To find the min $[P(\widetilde{TC}(\tilde{Q}, B))]$, we have to find the derivative of of $P(\widetilde{TC}(\tilde{Q}, B))$ with respect to Q_1, Q_2, Q_3, Q_4 and equal to zero and solve Q_1, Q_2, Q_3, Q_4.

$$Q_1 = \sqrt{\frac{2\left(A_4d_4+B\pi_4d_4\right)+B^2(h_4+\hat{\pi}_4)}{h_1}}$$

$$Q_2 = \sqrt{\frac{2\left(2A_3d_3+2B\pi_3d_3\right)+2B^2(h_3+\hat{\pi}_3)}{2h_2}}$$

$$Q_3 = \sqrt{\frac{2\left(2A_2d_2+2B\pi_2d_2\right)+2B^2(h_2+\hat{\pi}_2)}{2h_3}}$$

$$Q_4 = \sqrt{\frac{2\left(A_1d_1+B\pi_1d_1\right)+B^2(h_1+\hat{\pi}_1)}{h_4}}$$

Step 2 : Convert the inequality constraint $Q_2 - Q_1 \geq 0$ into equality constraint $Q_2 - Q_1 = 0$ and optimize $P\left(\widetilde{TC}(\tilde{Q},B)\right)$ subject to $Q_2 - Q_1 = 0$ and by the Lagrangean Method.

$L(Q_1, Q_2, Q_3, Q_4, \lambda) = P(\widetilde{TC}(\tilde{Q}, B)) - \lambda\left(Q_2 - Q_1\right)$. Taking the partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda)$ with respect to Q_1, Q_2, Q_3, Q_4 and λ to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda)$ and let all the partial derivatives equal to zero and solve Q_1, Q_2, Q_3, Q_4.

Step 3 : Optimize $P(\widetilde{TC}(\tilde{Q}, B))$ subject to $Q_2 - Q_1 = 0$ and $Q_3 - Q_2 = 0$ by the Lagrangean Method. Then $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2) = P(\widetilde{TC}(\tilde{Q}, B)) - \lambda_1\left(Q_2 - Q_1\right) - \lambda_2\left(Q_3 - Q_2\right)$.

The partial derivatives $\frac{\partial L}{\partial Q_1}$, $\frac{\partial L}{\partial Q_2}$, $\frac{\partial L}{\partial Q_3}$, $\frac{\partial L}{\partial Q_4}$, $\frac{\partial L}{\partial \lambda_1}$, $\frac{\partial L}{\partial \lambda_2}$ equal to zero and solve Q_1, Q_2, Q_3, Q_4.

Step 4 : Optimize $P(\widetilde{TC}(\tilde{Q}, B))$ subject to $Q_2 - Q_1 = 0$, $Q_3 - Q_2 = 0$ and $Q_4 - Q_3 = 0$ by the Lagrangean Method. Then $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_2) = P(\widetilde{TC}(\tilde{Q}, B)) - \lambda_1\left(Q_2 - Q_1\right) - \lambda_2\left(Q_3 - Q_2\right) - \lambda_3\left(Q_4 - Q_3\right)$.

The partial derivatives $\frac{\partial L}{\partial Q_1}, \frac{\partial L}{\partial Q_2}, \frac{\partial L}{\partial Q_3}, \frac{\partial L}{\partial Q_4}, \frac{\partial L}{\partial \lambda_1}, \frac{\partial L}{\partial \lambda_2}, \frac{\partial L}{\partial \lambda_3}$ equal to zero and solve Q_1, Q_2, Q_3 and Q_4.

$Q^* = Q_1 = Q_2 = Q_3 = Q_4$

$$= \sqrt{\frac{2(A_4d_4+2A_3d_3+2A_2d_2+A_1d_1)+2\left(B\pi_4d_4+2B\pi_3d_3+2B\pi_2d_2+B\pi_1d_1\right) + B^2\left[(h_4+\hat{\pi}_4)+2(h_3+\hat{\pi}_3)+2(h_2+\hat{\pi}_2)+(h_1+\hat{\pi}_1)\right]}{h_1+2h_2+2h_3+h_4}}$$

5. Numerical Examples

Consider an inventory system with the following characteristics.

$d = 250, A = 100, C = 10, h = 2, \pi = 0.5, \hat{\pi} = 0.4$

$Q^* = 361.20, B^* = 248.62, TC(Q, B) = 2724.56.$

Suppose Fuzzy annual demand is "more or less than 20"

$\tilde{d} = (d_1, d_2, d_3, d_4) = (220, 240, 260, 280)$

Fuzzy unit product cost rate is "more or less than 10"

$\tilde{C} = (C_1, C_2, C_3, C_4) = (8, 9, 11, 12)$

Fuzzy ordering cost is "more or less than 100"

$\tilde{A} = (A_1, A_2, A_3, A_4) = (80, 90, 110, 120)$

Fuzzy unit holding cost is "more or less than 2"

$\tilde{h} = (h_1, h_2, h_3, h_4) = (1.8, 1.9, 2.1, 2.2)$

Fuzzy unit backorder cost per time unit is "more or less than 0.4"

$\hat{\pi} = \left(\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4\right) = (0.2, 0.3, 0.5, 0.7)$

Fuzzy unit backorder cost (fixed) is "more or less than 0.5"

$\pi = \left(\pi_1, \pi_2, \pi_3, \pi_4\right) = (0.3, 0.4, 0.6, 0.7)$

Fuzzy order quantity

$\tilde{Q} = (Q_1, Q_2, Q_3, Q_4)$ with $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$.

Fuzzy backorder level

$\tilde{B} = (B_1, B_2, B_3, B_4)$ with $0 < B_1 \leq B_2 \leq B_3 \leq B_4$.

$Q^* = 356.35$

$B^* = 242.08$

Optimal total inventory cost function for the EOQ with backorders,

$$TC^*(Q, B) = (1803.2, 2526.49, 3200.9, 3786.09)$$

6. Conclusion

This paper presents two fuzzy EOQ model with backorders J.T. Teng [17], S.K. Goyal [8] and minimizing the total inventory cost function for the EOQ with backorders. In the first model, Order Quantity (Q) is treated as a fixed constant. In the second model (B) U.K. Gupta [9] is treated as a fixed constant and other notations are represented by fuzzy numbers. For each fuzzy model, a method of defuzzification, graded mean integration representation is applied to find estimate of total inventory cost function for the EOQ with backorders.

References

[1] A. Banerjee, A joint economic lotsize model for purchase and vendor, Decision Sciences 17 (1986) 292-311.

[2] L.F. Cardenas – Barron, A simple method to compute economic order quantities. Some observations, Appl. Math. Model 34(6) (2010) 1684-1688.

[3] S. Chand, J. Ward, A note on economic order quantity under conditions of permissible delay in payments. Computers and Operations Research 25 (1987) 49-52.

[4] S.H. Chen, Operations on fuzzy numbers with function principle, Tamkong J. of Management Sciences, 6(1) (1985) 13-26.

[5] S.H. Chen, C.H. Hsieh, Graded mean integration representations of generalized fuzzy numbers. J. of Chinese Fuzzy Systems 5 (1999) 1-7.

[6] S.C. Chang, Fuzzy production inventory for fuzzy production quantity with triangular fuzzy number, Fuzzy sets and systems 107 (1999), 37-57.

[7] R.W. Crubbstro, A. Erdem, The EOQ with backlogging derived with derivatives. Int. J. Prod. Econ. 59(1-3) (1999), 529-530.

[8] S.K. Goyal, Economic order quantity under conditions of permissible delay in payments. J. of Operational Research 41 (1985) 261-269.

[9] U.K. Gupta, A comment on economic order quantity under conditions of permissible delay in payments. J. of Operational Research Society 39 (1988) 322-323.

[10] F.W. Harris, How many parts to make at once, Factory, the Magazine of Management 10(1913), 135-136.152 Reprinted in: Operations Research 38 (1990) 947-950.

[11] Y.F. Huang, Optimal retailer's ordering policies in the EOQ model under trade credit financing. J. of Operational Research Society 54 (2003) 1011-1015.

[12] Leopoldo Edwardo Cardenas – Barron, The derivation of EOQ inventory models with two backorders costs using analytic geometry and algebra, Appl. Modell, (2010).

[13] K.S. Park, Fuzzy set theoretic interpretation of economic order quantity. IEEE. Transactions on Systems, Man, Cyberatics SMC, 17 (1987) 1082-1084.

[14] R.J. Ronald, G.K. Yang, P. Chu, Technical note the EOQ and EPQ models with shortages derived without derivatives, Int. J. Prod. Econ., (2004), 197-200.

[15] G.P. Sphicas, EOQ and EPQ with linear and fixed backorders costs: two cases identified and models analysed without Calculus, Int. J. Prod. Econ,. 100(1) (2006), 59-64.

[16] H.A. Taha, Operations Research, Prentice-Hall Englewood Cliffs, NJ, USA, (1997) 753-777.

[17] J.T. Teng, On the economic order quantity order conditions of permissible delay in payments. J. of Operation Research Society 53 (2002) 915-918.