Fuzzy Node Fuzzy Graph and its Cluster Analysis

Dr.E.Chandrasekaran
Associate Professor in Mathematics, Presidency College, Chennai. Tamilnadu, India.
e-mail : e_chandrasekaran@yahoo.com

N.Sathyaseelan
Assistant Professor in Mathematics, T.K Government Arts College, Vriddhachalam – 606 001.
e-mail : nseelan83@gmail.com

ABSTRACT

We could generally analyze inexact information efficiently and investigate the fuzzy relation by applying the fuzzy graph theory. We would extend the fuzzy graph theory and propose a fuzzy node fuzzy graph to the crisp node fuzzy graph by applying T-norms. In this paper, we would discuss about Fuzzy node fuzzy graph,fuzzy partial graph,Transformation from the fuzzy node fuzzy graph to the crisp node fuzzy graph, New T-norms, "sathyam product", Cluster structure analysis of fuzzy node fuzzy graph, And Global structure analysis of the optimal fuzzy graph G_λ in the fuzzy sequence $\{ G_\lambda \}$. By using the fuzzy node fuzzy graph theory and this new T-norm, we would clarify the relational structure of fuzzy information and by using the decision of an optimal level on a partition tree. We could analyze the clustering relation among node.

Keywords: crisp node fuzzy graph, fuzzy partial graph, fuzzy node fuzzy graph and quasi logical product.

1. INTRODUCTION

Fuzzy graph theory was introduced by Azriel Rosenfield in 1975. Though it is very young it has been growing fast and has numerous applications in various fields. In this simplest form a graph consists of set of elements (or nodes) and a set of ordered and unordered pairs of nodes (or edges). We would extend the fuzzy graph theory and propose a fuzzy node fuzzy graph. Since a fuzzy node fuzzy graph is complicated to analyse, we would transform it to a simplest fuzzy graph by using T-norms family. A graph is defined by a pair G = (V,F) where $V=\{v_1, v_2, \ldots v_n\}$ is a finite set of vertices and F, a collection of edges that happen to connect these vertices. The edge set F graphically represented as V × V. The concept of fuzzy graph is the fuzzyfication of the crisp graphs using fuzzy sets. In this paper we introduce fuzzy node fuzzy graph, crisp node fuzzy graph, fuzzy partial graph and new T-norms (Triangular norms). Also we study some properties of fuzzy node fuzzy graphs and examine whether they hold for crisp node fuzzy graphs.

2. PRELIMINARIES

2.1. FUZZY NODE FUZZY GRAPH

Definition – 1: Crisp Node Fuzzy Graph

The crisp node fuzzy graph G is defined by

$$G = (V,F) : V = \{v_i\},\ F = \{f_{ij}\},\ 0 \le f_{ij} \le 1.$$

Where V is the set of nodes and F is an $n \times n$ matrix whose (i,j) component f_{ij} is a fuzziness of the arc from the node v_i to the node v_j.

Definition -2: Fuzzy Partial Graph

If the fuzzy graph G = (V,F) and the fuzzy graph G' = (V,F') satisfies:

$F = (f_{ij})$, $F' = (f'_{ij})$, $f_{ij} \leq f'_{ij}$.

We call fuzzy graph G' as fuzzy partial graph of fuzzy graph g and denote $G' \pi G$.

Definition -3: Fuzzy Node Fuzzy Graph

A fuzzy node fuzzy graph G is defined by,

$G = (V, F) = \{v_i / u_i\}, Y = (y_{ij}), 0 \leq u_i \leq 1, 0 \leq y_{ij} \leq 1$

Where V is the set of the nodes and the fuzziness u_i is a fuzziness of the node v_i. Y is an $n \times n$ matrix whose (i, j) component y_{ij} is a fuzziness of the arc from the node v_i to the node v_j.

A fuzzy node fuzzy graph is characterized by the fuzziness of the nodes and the fuzziness of the arcs. Therefore, the structure of a fuzzy node fuzzy graph is usually very complicated. Then, it should be interesting to transform a fuzzy node fuzzy graph to a crisp node fuzzy graph and we present a method to transform a fuzzy node fuzzy graph to a crisp node fuzzy graph.

Definition – 4: Transformation from Fuzzy Node Fuzzy Graph to Crisp Node Fuzzy Graph

A fuzzy node fuzzy graph $G=(V, Y)$ can be transformed to a crisp node fuzzy graph $G=(V,F)$ by the following method: Let

$$G = (V, F) : V = \{v_i\}, F = (f_{ij}), f_{ij} = T(u_i, y_{ij})$$

Where the fuzziness f_{ij} of the arc from the node v_i to the node v_j could be defined by applying T-norms.

This fuzzy node fuzzy graph analysis would be applied to the sociometry analysis, the instruction structure analysis and so on.

2.2 T-norm and its family

T-norm (Triangular Norm) is a binary operation that is defined by the following:

Definition - 5 : T-Norm is a binary operation

$$p, q \in [0, 1] \rightarrow (p, q) \in [0, 1]$$

Satisfying the following properties.

- Commutativity:

  $T(p,q) = T(q,p)$

- Associativity :

  $T(p,T(q,r)) = T(T(p,q),r)$

- Monotonicity :

  $p \le q,\ r \le s \Rightarrow T(p,r) \le T(q,s)$

- Boundary conditions:

  $T(p,0) = 0,\ T(p,1) = p$

  The typical T-norms are the followings:

- Logical product: $T_L(p, q) = p \wedge q$
- Algebraic product: $T_A(p, q) = pq$
- Multi-valued product (Lukaszewicz product) : $T_M(p, q) = (p + q - 1) \vee 0$
- Drastic Product: $T_D(p,q) = \begin{cases} 0, & p \vee q < 1 \\ p \wedge q, & p \vee q < 1 \end{cases}$

  Here, "$a \wedge b$" means **"min (a, b)"**, "$a \vee b$" means **"max (a, b)"**.

Definition - 6 : Order of T-Norms

For two T-norms T_α and T_β, if a relation

$T_\alpha(p, q) \le T_\beta(p, q), (p, q) \in [0, 1]^2$

Holds, we denote it by $T_\alpha \le T_\beta$.

For any T-norm T, a relation $T_D \le T \le T_L$ always holds.

Definition - 7: T-Norm Family

For any $\lambda \in [a, b]$, when T_λ is T-norm, then we say that $\{T_\lambda\}$ is T-norm family that connects with T_a and T_b.

The typical T-norm families are

- Dubois product:

  $$T_\lambda(p, q) = \frac{pq}{p \vee q \vee \lambda},\quad \lambda \in [0, 1]$$

- Weber product:

  $$T_\lambda(p, q) = 0 \vee \{(1 + \lambda)(p + q - 1) - \lambda pq\},\quad \lambda \ge -1$$

- Schweizer Product

$$T_\lambda(p,q) = \sqrt[\lambda]{0 \vee (p^\lambda + q^\lambda - 1)}, \lambda > 0$$

- Sathyam product (Quasi – logical product)

$$T_\lambda(p,q) = \begin{cases} 0, & p \vee q < 1-\lambda \\ p \wedge q, & p \vee q \geq 1-\lambda \end{cases}, \quad \lambda \in [0, 1] \text{ and so on.}$$

The Relation of typical T-norm families and Sathyam product could be illustrated in Fig. 2.1.

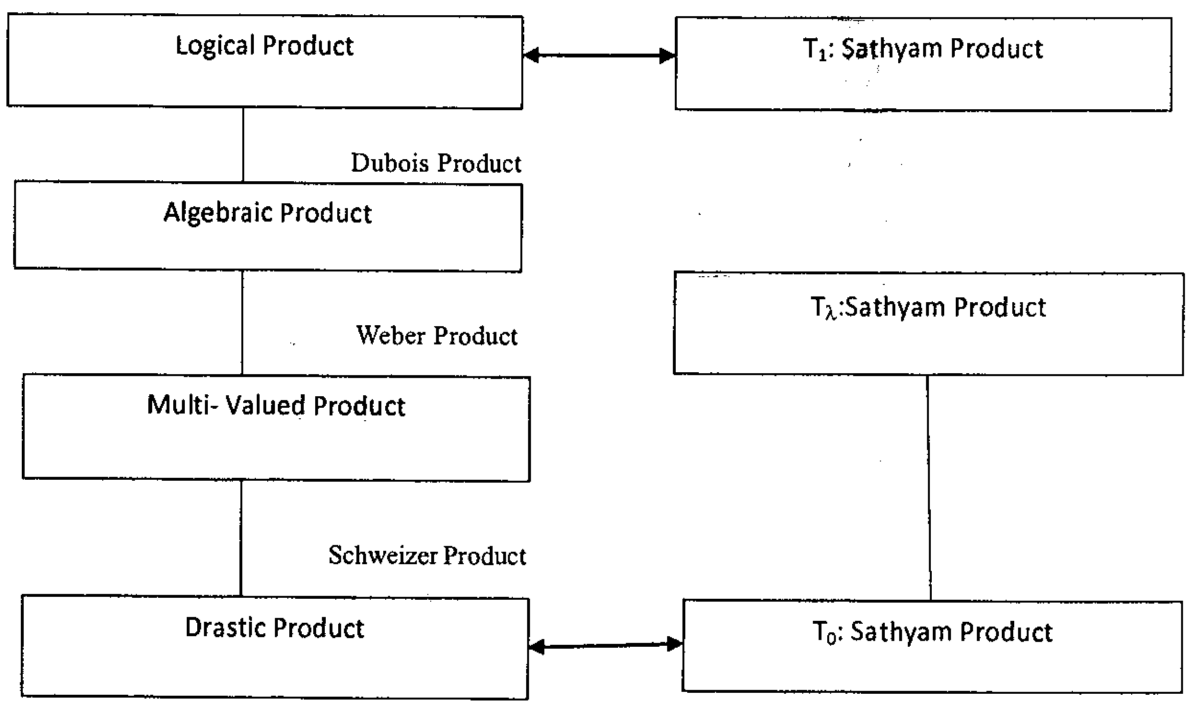

Fig. 2.1 Relation of typical T-norm families

Theorem - 1

The fuzzy graph series $\{G_\lambda\}$ is the fuzzy partial graph series by using the Sathyam product.

Proof

Let the fuzzy node fuzzy graph;

$$G=(V,Y): V=\{v_i(u_i)\}, Y=(y_{ij})$$

If $\lambda_1 < \lambda_2$, we obtained two crisp node fuzzy graphs;

$$G_{\lambda_1}=(V,F): V=\{v_i\}, F=(f_{ij}), f_{ij}=\begin{cases} 0 & ,u_i \vee y_{ij} < 1-\lambda_1 \\ u_i \wedge y_{ij} & ,u_i \vee y_{ij} \geq 1-\lambda_1 \end{cases}$$

$$G_{\lambda_2}=(V,F'): V=\{v_i\}, F'=(f'_{ij}), f'_{ij}=\begin{cases} 0 & ,u_i \vee y_{ij} < 1-\lambda_2 \\ u_i \wedge y_{ij} & ,u_i \vee y_{ij} \geq 1-\lambda_2 \end{cases}$$

Since the Sathyam product is monotonous, then $f_{ij} \leq f'_{ij}$.

$$\therefore\ G_{\lambda_1} \pi\, G_{\lambda_2}.$$

3. CLUSTERING STRUCTURE ANALYSIS OF FUZZY NODE FUZZY GRAPH

3.1 Cluster Analysis of Fuzzy Graph

A fuzzy graph *G* is defined by

$$G=(V,Y), V=\{v_i\}, F=(f_{ij}), 0 \leq f_{ij} \leq 1, f_{ii}=1, 1 \leq i \leq n$$

Where f_{ij} is a fuzziness of the arc from the node v_i to the node v_j.

In order to analyze the similarity structure of nodes for a fuzzy graph G=(V,F), we use the symmetric relation matrix $S=(s_{ij})$.

A symmetric relation matrix *S* could be defined by using the arithmetic mean, the geometric mean, the harmonic mean and so on. Here, we define the symmetric relation matrix *S* by using the harmonic mean.

A symmetric relation matrix *S* is defined by

$$S=(s_{ij}), \frac{2}{s_{ij}}=\frac{1}{f_{ij}}+\frac{1}{f_{ji}}$$

Where $s_{ij}=0$ if $f_{ij} . f_{ij}=0$.

In order to analyze the clustering structure among nodes, we have its max-min transitive closure $\hat{S}=(\hat{s}_{ij})$ which is computed by $\hat{S}=S^n$.

After that, we define the c-cut matrix S_c as follows:

$$S_c = (s_{ij}^c) = s_{ij}^c = \begin{cases} 0 & (\hat{s}_{ij} \geq c) \\ 0 & (\hat{s}_{ij} < c) \end{cases}, 0 \leq c \leq 1$$

From the matrix S_c, we define the cluster $CL_{sc}(i)$.

$$J_c(i) = \{j \mid s_{ij}^c = 1, 1 \leq j \leq n\}$$

$$CL_{Sc}(i) = \{x_j \mid j \in J_c(i)\}$$

The cluster $CL_{Sc(i)}$ = gives an equivalence relation among nodes.

Hence, we can construct the partition tree by changing the level *c* of the *c*-cut matrix which represents the clustering situation of nodes in a fuzzy graph.

For example, if a fuzzy matrix F is in Fig. 3.1, then we have obtained the symmetric matrix S in Fig. 3.2 and the transitive closure $\hat{S} = (\hat{s}_{ij})$ is in Fig. 3.3, by changing the value of c, we have the c-cut matrix S_c in Fig. 3.4 - 3.8.

	1	2	3	4	5	6	7	8
1	1	-	0.67	0.33	0.50	0.17	0.83	1.00
2	-	1	1.00	0.83	0.67	-	0.33	0.50
3	1.00	0.33	1	0.17	0.50	-	0.67	0.83
4	-	0.83	1.00	1	0.33	0.17	0.67	0.50
5	0.83	0.50	0.67	-	1	1.00	0.33	0.17
6	-	-	-	-	1.00	1	-	-
7	0.83	0.33	0.67	0.50	0.17	-	1	1.00
8	0.67	0.17	1.00	0.50	0.33	-	0.83	1

Fig. 3.1 Fuzzy Matrix *F*

	1	2	3	4	5	6	7	8
1	1	-	0.80	-	0.62	-	0.83	0.80
2	-	1	0.50	0.83	0.57	-	0.33	0.25
3	0.80	0.50	1	0.29	0.57	-	0.67	0.91
4	-	0.83	0.29	1	-	-	0.57	0.50
5	0.62	0.57	0.57	-	1	1.00	0.22	0.22
6	-	-	-	-	1.00	1	-	-
7	0.83	0.33	0.67	0.57	0.22	-	1	0.91
8	0.80	0.25	0.91	0.50	0.22	-	0.91	1

Fig. 3.2 Symmetric Matrix *S*

	1	2	3	4	5	6	7	8
1	1	0.57	0.83	0.57	0.62	0.62	0.83	0.83
2	0.57	1	0.57	0.83	0.57	0.57	0.57	0.57
3	0.83	0.57	1	0.57	0.62	0.62	0.91	0.91
4	0.57	0.83	0.57	1	0.57	0.57	0.57	0.57
5	0.62	0.57	0.62	0.57	1	1.00	0.62	0.62
6	0.62	0.57	0.62	0.57	1.00	1	0.62	0.62
7	0.83	0.57	0.91	0.57	0.62	0.62	1	0.91
8	0.83	0.57	0.91	0.57	0.62	0.62	0.91	1

Fig. 3.3 Transitive Closure $\hat{S} = (\hat{s}_{ij})$

	1	2	3	4	5	6	7	8
1	1							
2		1						
3			1					
4				1				
5					1	1		
6					1	1		
7							1	
8								1

Figure 3.4 c-cut Matrix S_c ($c \in (0.91, 1.00]$)

	1	2	3	4	5	6	7	8
1	1							
2		1						
3			1				1	1
4				1				
5					1	1		
6					1	1		
7			1				1	1
8			1				1	1

Fig. 3.5 c-cut Matrix S_c ($c \in$ (0.83, 0.91])

	1	2	3	4	5	6	7	8
1	1		1				1	1
2		1		1				
3	1		1				1	1
4		1		1				
5					1	1		
6					1	1		
7	1		1				1	1
8	1		1				1	1

Fig. 3.6 c-cut Matrix S_c ($c \in$ (0.62, 0.83])

	1	2	3	4	5	6	7	8
1	1		1		1	1	1	1
2		1		1				
3	1		1		1	1	1	1
4		1		1				
5	1		1		1	1	1	1
6	1		1		1	1	1	1
7	1		1		1	1	1	1
8	1		1		1	1	1	1

Fig. 3.7 c-cut Matrix S_c ($c \in$ (0.57, 0.62])

	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1

Fig.3.8c-cut Matrix S_c ($c \in$ [0.00, 0.57])

From the *c*-cut matrix *Sc,* we have constructed the partition tree For example, the partition tree cluster were merged at the value of c=0.83, we say that merged at cluster level $R_{0.83.}$

Concerning the cluster analysis of the fuzzy graph, it is important to decide the optimal level of fuzzy clustering as to the partition tree. Here, we would adopt the decision method of the optimal cut level c_0 by applying the fuzzy decision.

3.2 Global Structure Analysis Fuzzy Node Fuzzy Graph

Concerning the transformation from a fuzzy node fuzzy graph $G = (V, Y)$ to a fuzzy graph $G_\lambda = (V, F_\lambda)$, we could analyze a fuzzy node fuzzy graph by using Sathyam product T_λ.

By changing the parametersλ, a sequence $\{G_\lambda\}$ of fuzzy graph is composed.

Here, we would choose the optimal fuzzy graph G_{λ_0} from this fuzzy graph sequence $\{G_\lambda\}$.

Then, we would discuss the decision method of the optimal parameter λ_0.

In order to decide the optimal fuzzy graph , we would define two functions $d(\lambda)$ and $e(\lambda)$ as follows :

Definition - 8 : Distance Function d(λ) and Connectivity function e(λ)

$$d(\lambda) = d(F_\lambda, S_{c_0}) = \frac{1}{n^2 - n}\sum_{i=1}^{n}\sum_{j=1}^{n} | f_{ij} - s_{ij}^{c_0} |$$

$$e(\lambda) = e(F_\lambda) = \frac{\gamma(F_\lambda)}{n^2 - n}$$

Where, $\gamma(F_\lambda) = \#(\Gamma_\lambda), \Gamma_\lambda = \{f_{ij}^\lambda \in F_\lambda \mid f_{ij}^\lambda > 0\}$

And F_λ, S_{c_0} are given as follows.

$$F_\lambda = (f_{ij}^\lambda),$$

$$f_{ij}^\lambda = \begin{cases} 0 & , u_i \vee y_{ij} < 1-\lambda \\ u_i \wedge y_{ij} & , u_i \vee y_{ij} \geq 1-\lambda \end{cases}$$

$$S_{c_0} = (s_{ij}^{c_0}), s_{ij}^{c_0} = \begin{cases} 1 & (\hat{s}_{ij} \geq c_0) \\ 0 & (\hat{s}_{ij} < c_0) \end{cases}$$

Here, $d(\lambda)$ evaluates the feature between the fuzzy node fuzzy graph $G_\lambda = (V, F_\lambda)$ and the optimal clustering cut level c_0. If the value of $d(\lambda)$ is large, then G_λ reasonably shows the feature of the clustering level c_0.

On the other hand, $e(\lambda)$ evaluates the connectivity information of G_λ . If the value of $e(\lambda)$ is small, then G_λ reasonably shows the feature of the connectivity information.

Here, we would normalize the values of $d(\lambda)$ and $e(\lambda)$ respectively, and define $f_d(\lambda)$ and $f_e(\lambda)$ as follows:

Definition - 9 : Fuzzy Distance Function $f_d(\lambda)$ and Fuzzy Connectivity Function (λ)

$$f_d(\lambda) = \frac{d_M - d(\lambda)}{d_M - d_m}, f_e(\lambda) = \frac{e(\lambda) - e_m}{e_M - e_m}$$

Where,

$$d_M = \vee_{\lambda\in[0,1]}\{d(\lambda)\}, d_m = \wedge_{\lambda\in[0,1]}\{d(\lambda)\},$$

$$e_M = \vee_{\lambda\in[0,1]}\{e(\lambda)\} \text{ and } e_m = \wedge_{\lambda\in[0,1]}\{e(\lambda)\},$$

By applying the maximal decision of the fuzzy decision, we could reasonably find the optimal value λ_0 concerning the sequence $\{G_\lambda\}$.

Definition - 10 : Decision of optimal value λ_0.

$$f_m(\lambda) = f_d(\lambda) \wedge f_e(\lambda)$$

$$\lambda_0 = \wedge \{\lambda : f_m(\lambda) = \vee_{\lambda\in[0,1]} f_m(x)\}$$

According to this decision method, we would obtained the optimal Fuzzy graph G_{λ_0} .

4. CONCLUSION

In this paper, we have mainly discussed the fuzzy node fuzzy graph, new T-norm "Sathyam Product",Decision analysis of the optimal fuzzy graph G_{λ_0} in the fuzzy graph sequence $\{G_\lambda\}$. We have illustrated the practical effectiveness of the analysis method. This method is also available for the instruction/ cognition analysis in education, the opinion poll in psychology and so on.

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