b-CHROMATIC NUMBER FOR SOME GRAPHS

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Abstract

A b-coloring of a graph G is a proper vertex coloring of G such that each color class contains a vertex that has at least one neighbor in every other color class and b-chromatic number of a graph G is the largest integer $\phi(G)$ for which G has a b-coloring with $\phi(G)$ colors. In this paper, we have obtained the b-chromatic number of the graphs $P_n \times P_m$, $C_n \times C_m$, W_n, $S_2(G)$, T_n and graphs obtained by duplicating each vertex of path and cycle by an edge .

Keywords. b-coloring, adjacency Graph.

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1. INTRODUCTION

Let G be a graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. A proper k-coloring of graph G is a function C defined on $V(G)$

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onto a set of colors $C = \{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colors. For every $i, 1 \leq i \leq k$, the set C_i is an independent set of vertices which is called a color class. Let P_n be a path graph with n vertices and $n-1$ edges. Let C_n be a cycle with n vertices and n edges. The graph $S_2(G)$ is a graph obtained from G by subdividing each edge of G by a new vertex. Corona product or simply corona of graphs G_1 and G_2 is a graph which is the disjoint union of one copy of G_1 and $|v_1|$ copies of G_2 ($|v_1|$ is number of vertices of G_1) in which each vertex copy of G_1 is connected to all vertices of separate copy of G_2. The Cartesian product of two graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u, u') and (v, v') are adjacent in $G_1 \times G_2$ if and only if either $u = v$ and u' is adjacent with v' or $u' = v'$ and u is adjacent with v. The b-chromatic number of a graph was introduced by R. W. Irving and D. F. Manlove when considering minimal proper coloring with respect to a partial order defined on the set of all partition of vertices of graph. The b-chromatic number of a graph G, denoted by $\phi(G)$, is the largest integer t such that there exists a proper coloring for G with t colors in which every color class contains at least one vertex adjacent to some vertex in all the other color classes and such a coloring is called b-coloring. So many authors have studied on b-chromatic number. Motivated by these works we have found b-chromatic number for the graphs $P_m \times P_n$, $C_n \times C_m$, W_n, $S_2(G)$, T_n and graphs obtained by duplicating each vertex of path and cycle by an edge.

2. MAIN RESULTS

Proposition 2.1. *For any positive integer n.*

$$\phi(P_2 \times P_n) = \begin{cases} 4 & \text{if } n \geq 4 \\ 2 & \text{if } n = 1, 2, 3. \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of length n-1 respectively in $P_2 \times P_n$.

Case 1. $n \geq 4$.

Assign the colors $1, 2, 3, 4, 2, 1, 2, 1, \ldots$ and $3, 4, 1, 2, 1, 2, 1, 2, \ldots$ for the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$. Then in each color class, there is a vertex whose all neighbors are in the remaining color classes. Since $\Delta(P_2 \times P_n) = 3$ and

$\phi(G) \leq \Delta(G)+1$, $\phi(G)=4$.

Case 2. $n=3$.

If u_2 and v_2 are colored by 1 and 2 (or 2 and 3 or 1 and 3), then 3 (or 1 or 2) may be given as color for either u_1 or u_3 or v_1 or v_3 which does not give a proper coloring and hence $\phi(P_2 \times P_3) = 2$.

Case 3. $n=2$.

In this case, $P_2 \times P_2$ is C_4 which is of b-chromatic number 2.

Case 4. $n=1$.

In this case, $P_2 \times P_1$ is P_2 which is of b-chromatic number 2. □

Theorem 2.2. *If m and n are positive integers such that $m \geq 4$ and $n \geq 5$, then $\phi(P_m \times P_n) = 5$.*

Proof. Let $v_{i,j}$, $1 \leq i \leq m$ and $1 \leq j \leq n$ be the vertices of $P_m \times P_n$. The first five vertices namely $v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}$ and $v_{1,5}$ are colored by 1,2,3,4 and 5 respectively. By taking $(i-1)^{th}$ power of the permutation $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$, we may obtain the colors of first five vertices of each $(i-1)^{th}$ row, $2 \leq i \leq m$. So the colors of the vertices in the first four rows and five columns are respectively.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & \mathbf{4} & 5 & 1 & 2 \\ 5 & \mathbf{1} & \mathbf{2} & \mathbf{3} & 4 \\ 2 & 3 & 4 & 5 & 1 \end{array}$$

From the above array of numbers which are in bold, we observe that there exits a vertex in each color class which is adjacent to at least one vertex in each other color classes. The colors of the remaining column vertices are given by

$$C(v_{i,j}) = \begin{cases} C(v_{i+1,j-1}) & \text{if } 5 \leq j \leq n \text{ and } i \neq m \\ C(v_{i,j-1}) & \text{if } 5 \leq j \leq n. \end{cases}$$

and it gives a proper coloring for $P_m \times P_n$.

Also $\phi(P_m \times P_n) \leq \Delta(P_m \times P_n) + 1 = 5$. Therefore, $\phi(P_m \times P_n) = 5$. □

When $m=3$ and $n \geq 7$, $\phi(P_3 \times P_n) = 5$ as in the proof of Theorem 2.2. When $m=3$ and $3 \leq n \leq 6$, b-coloring with 5 colors is not possible since the number of vertices of degree 4 is less than 5. In this case, the b-coloring with 4 colors are given in Figure 1.

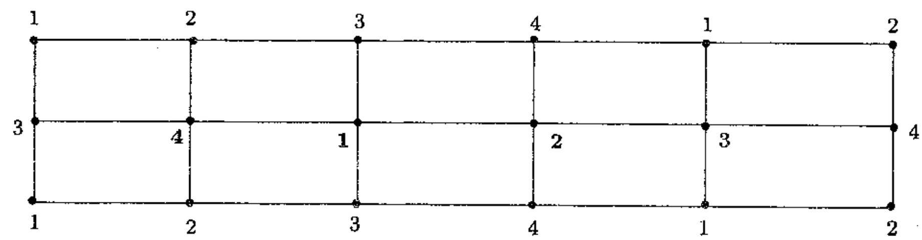

Figure 1. $P_3 \times P_6$

Theorem 2.3. *$\phi(C_m \times P_n) = 5$, where $m \geq 4$ and $n \neq 7t + 1$ t being positive integer.*

Proof. When $m \geq 4$ and $n \neq 7t+1, t$ is a positive integer, as in the coloring given in Theorem 2.2, the b-coloring with 5 colors is possible and hence $\phi(C_m \times P_n) = 5$. When $n = 7t + 1$, fill the colors for the first five vertices of each row upto $n - 1$ as in Theorem 2.2 and for the vertices in n^{th} row, the first five vertices will be colored as 2, 3, 4, 5, 1 respectively. The colors of the remaining column vertices are given by

$$C(v_{i,j}) = \begin{cases} C(v_{i+1,j-1}) & \text{if } 5 \leq j \leq n \text{ and } i \neq m \\ C(v_{1,j-1}) & \text{if } 5 \leq j \leq n \text{ and } i = m. \end{cases}$$

and it gives a proper coloring for $C_m \times P_n$.

When $m = 3$ and $n \geq 7$, it is possible for giving b-coloring with 5 colors as in $P_3 \times P_n$ and hence $\phi(C_3 \times P_n) = 5$. When $m = 3$ and $3 \leq n \leq 6$, b-coloring with 5-colors is not possible since the number of vertices with degree 4 is less than 5. In this case, the b-coloring with 4-colours are given in Figure 2.

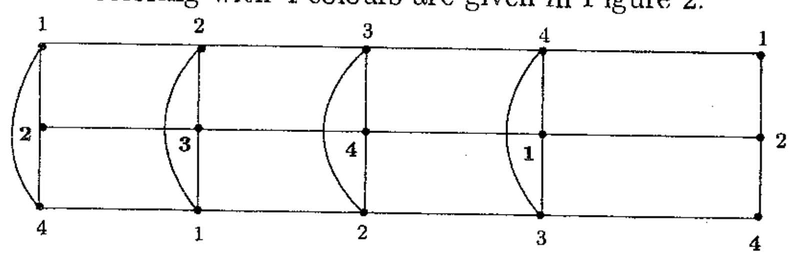

Figure 2. $C_3 \times P_5$

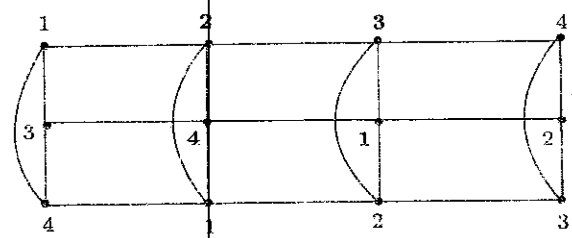

Figure 3. $C_3 \times P_4$

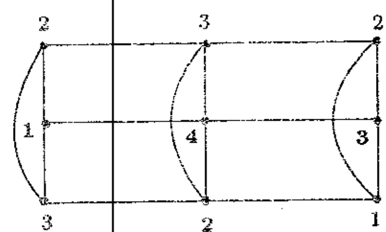

Figure 4. $C_3 \times P_3$

Proposition 2.4. *The b-chromatic number for a wheel graph is*

$$\phi(W_n) = \begin{cases} 3, & n = 5 \\ 4, & n \geq 4 \text{ and } n \neq 5. \end{cases}$$

Proof. Let v_1 be the central vertex and $v_2, v_3, \ldots, v_n$ be the vertices on the cycle in W_n. Since the central vertex v_1 is adjacent to all the vertices on the cycle, v_1 is colored by distinguished color. When n is even, 3 colors are needed to color these vertices on cycle as the number of vertices on the cycle is odd. Therefore the vertex class will be $C_1 = \{v_1\}, C_2 = \{v_2, v_4, v_6, \ldots, v_{n-2}\}, C_3 = \{v_3, v_5, \ldots, v_{n-1}\}$ and $C_4 = \{v_n\}$. In C_1, v_1 is adjacent to each and every vertex of remaining color classes. In C_2, there is a vertex v_2 adjacent to v_1 in C_1, v_3 in C_3 and v_n in C_4. In C_3, there is a vertex v_{n-1} adjacent to v_1 in C_1 ,v_{n-2} in C_2 and v_n in C_4. In C_4, v_n is adjacent to v_1 in C_1, v_2 in C_2 and v_{n-1} in C_3. Since only one vertex is of degree $n-1$ and all the remaining is of degree 3. $\phi(W_n) = 4$. When $n \neq 5$ is odd, by taking the color classes $C_1 = \{v_1\}, C_2 = \{v_3, v_6, v_8, v_{10}, \ldots, v_{n-1}\}, C_3 = \{v_4, v_7, v_9, \ldots, v_n\}$ and $C_4 = \{v_2, v_5\}$, the graph W_n has a proper b-coloring with 4 colors. Since no vertex other than the central vertex in the cycle is of degree more than 3, W_n cannot have b-coloring with more than 4 colors. Therefore $\phi(W_n) = 4$

when $n \neq 5$. In W_5, by assigning the color 1 for the central vertex and 2, 3, 4 for any of the three vertices on the cycle, no one of the vertex in the color class C_3 and C_4, C_2 and C_4 or C_2 and C_3 is adjacent to at least one vertex of each other classes, when 2 (3 or 4) is colored in the remaining vertex. Hence $\phi(W_n) < 4$. The b-coloring of W_5 with 3 colors is shown in Figure 5.

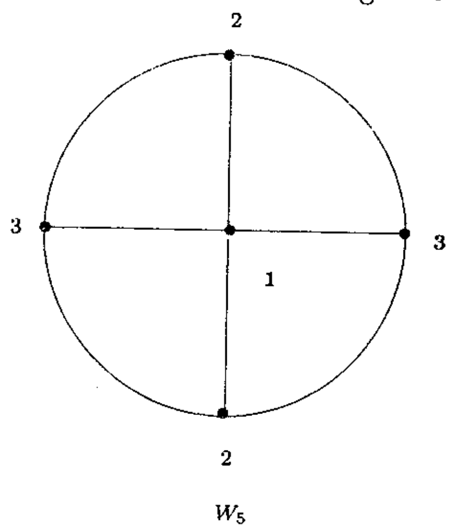

Figure 5

Proposition 2.5. *For the triangular belt graph* $T_n, n \geq 5$, $\phi(T_n) = 5$.

Proof. Since $\Delta(T_n) = 4$ for $n \geq 3$, the maximum value for the b-chromatic number is 5. Assume $n \geq 5$. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of length $n-1$. The edge set of T_n is $\{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. We color the vertices of u_i as a sequence $\{4, 1, 2, 3, 5, 4, 1, 2, 3, 5, \ldots\}$ and the vertices v_i as a sequence $\{2, 3, 5, 4, 1, 2, 3, 5, 4, 1, \ldots\}$. The vertices in the color classes of the colors 1, 2, 3, 4 and 5 are respectively $C_1 = \{u_2, u_7, u_{12}, \ldots, v_5, v_{10}, v_{15}, \ldots\}$, $C_2 = \{u_3, u_8, u_{13}, \ldots, v_1, v_6, v_{11}, \ldots\}$, $C_3 = \{u_4, u_9, u_{14}, \ldots, v_2, v_7, v_{12}, \ldots\}$, $C_4 = \{u_1, u_6, u_{11}, \ldots, v_4, v_9, v_{14}, \ldots\}$ and $C_5 = \{u_5, u_{10}, u_{15}, \ldots, v_3, v_8, v_{13}, \ldots\}$.

In the color classes C_1, C_2, C_3, C_4 and C_5, the vertices u_2,u_3,u_4,v_4 and v_3 are respective members in which they are adjacent to at least one member of the other color classes. Hence $\phi(T_n) = 5$ for $n \geq 5$. In T_4, since the number of vertices with degree 4 is less than 5, $\phi(T_4) \leq 4$. The b-coloring with 4 colors is given in Figure 6.

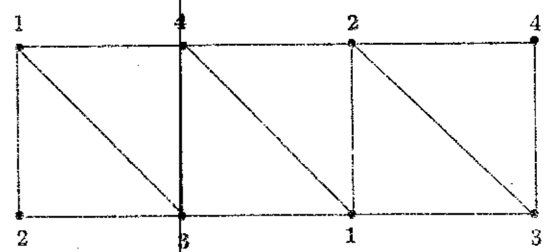

Figure 6. T_4

In T_3 also, the number of vertices with degree 4 is less than 5. The b-coloring with 4 colors for T_3 is given in Figure 7.

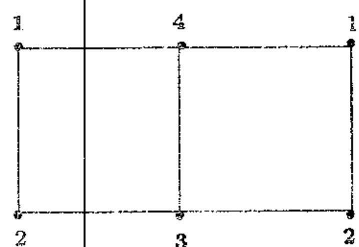

Figure 7. T_3

In T_2, $\Delta(T_2) = 3$ and number of vertices with degree 3 is less than 4. The b-coloring with 3 colors for T_2 is given below.

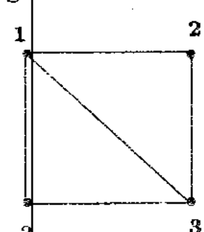

Figure 8. T_2

Theorem 2.6. *For any graph G, $\phi(S_2(G)) = \phi(G)$.*

Proof. By the definition of $S_2(G)$, $\Delta(S_2(G)) = \Delta(G)$. Let $\phi(G) = m$. Then there is a b-coloring with color classes $C_1, C_2, \ldots, C_m$ in which at least one vertex say v_i in C_i is adjacent to at least one member of the remaining $C_i' s, 1 \leq i \leq m$. If $v_iv_j \in E(G)$, then this edge is subdivided by two vertices x_i, y_j in $S_2(G)$. We may assign the color $C(v_i)$ to y_j and $C(v_j)$ to x_i. Hence the result follows.

Corollary 2.7. *If each edge of G is subdivided by even number of vertices, then the resultant graph have a b-chromatic number as much as b-chromatic number of G.*

Corollary 2.8. *If $\phi(G) \geq 3$, then any subdivision of G has a b-chromatic number as much as $\phi(G)$.*

Theorem 2.9. $\phi(P_n \circ mK_1) = \begin{cases} m+3 & \text{if } n \geq m+5 \\ m+2 & \text{if } m+2 \leq n \leq m+4 \\ m+1 & \text{if } n = m+1 \\ n & \text{if } 2 \leq n \leq m \\ 2 & \text{if } n = 1 \text{ and } n < m. \end{cases}$

Proof. In $P_n \circ mK_1$, let $u_1, u_2, \ldots, u_n$ be the vertices on the path of the length $n-1$ and $v_{i,1}, v_{i,2}, \ldots, v_{i,m}$ be the pendent vertices attached to u_i, $1 \leq i \leq n$.

Case 1. $n \geq m+5$.

Since $\Delta(P_n \circ mK_1) = m+2$ and there are at least $m+3$ vertices having degree $m+2$, $\phi(P_n \circ mK_1) \leq m+3$. We assign the colors $0, 1, 2, \ldots, m+2$ as follows: Color the vertices u_i by $i \ mod(m+3), 1 \leq i \leq n$. At each $i, 2 \leq i \leq n-1$, the set of vertices $\{v_{i,1}, v_{i,2}, \ldots, v_{i,m}\}$ can be colored by m different colors other than the colors assigned to u_{i-1}, u_i, u_{i+1}. The vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,m}$ are colored by m different colors other than the colors given to u_1 and u_2 and the vertices $v_{n,1}, v_{n,2}, \ldots, v_{n,m}$ are colored by m different colors other than the colors given to u_{n-1} and u_n. The vertices $u_2, u_3, \ldots, u_{m+4}$ are the members of the respective color classes of the colors $2, 3, 4, \ldots m+2, 0, 1$ in which they are adjacent to at least one vertex of each color class. Thus $\phi(P_n \circ mK_1) = m+3$.

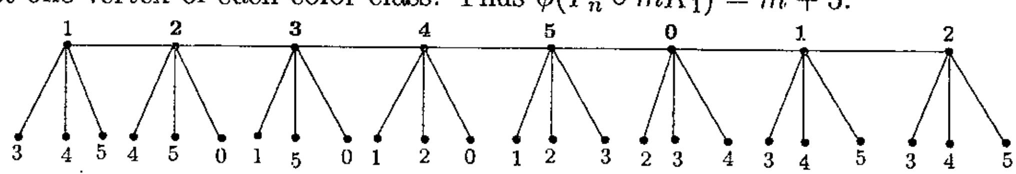

Figure 9. $P_8 \circ 3K_1$

Case 2. $m+2 \leq n \leq m+4$.

Since $\Delta(P_n \circ mK_1) = m+2$ and the number of vertices with degree $m+2$ is less than $m+3$, $\phi(P_n \circ mK_1) \leq m+2$. As in the previous case, color the pendent vertices by assign the color $i \ mod(m+2)$ to $u_i, 1 \leq i \leq n$.

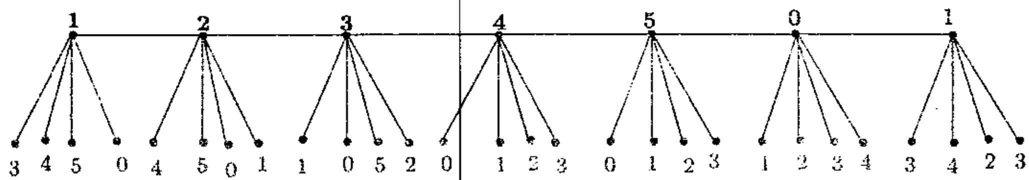

Figure 10. $P_7 \circ 4K_1$

The vertices $u_1, u_2, \ldots, u_{m+2}$ which are members of the respective color classes $1, 2, 3, \ldots, m+1, 0$, are adjacent to at least one member of the remaining color classes. Thus $\phi(P_n \circ mK_1) = m+2$.

Case 3. $n = m+1$.

In this case, the number of vertices with degree $\Delta(P_n \circ mK_1) = m+2$ is $m-1$. Therefore, $\phi(P_n \circ mK_1) < m+2$. We may color the pendent vertices as in Case 1 after coloring u_i by the color $i\ mod(m+1), 1 \leq i \leq n$. The vertices $u_1, u_2, \ldots, u_m, u_{m+1}$ which are the members of respective color classes of the colors $1, 2, 3, \ldots, m, 0$, are adjacent to at least one member of the remaining color classes. Thus $\phi(P_n \circ mK_1) = m+1$.

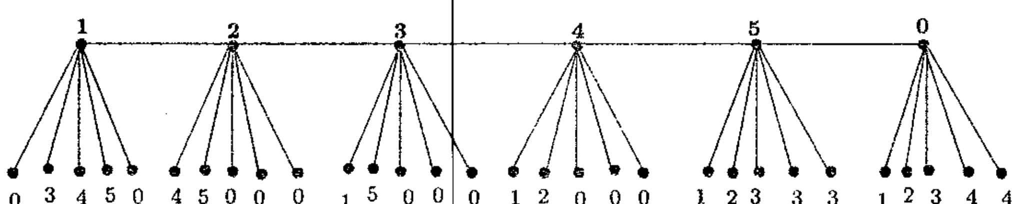

Figure 11. $P_6 \circ 5K_1$

Case 4. $2 \leq n \leq m$.

Since the number of vertices in the path is less than or equal to m, $\phi(P_n \circ mK_1) \leq n$. We assign the colors $0, 1, 2, \ldots, n-1$ to the vertices as follows: Color the vertices v_i by $i\ mod\ n$ and giving the colors to the pendent vertices as in Case 1 which implies that $\phi(P_n \circ mK_1) = n$.

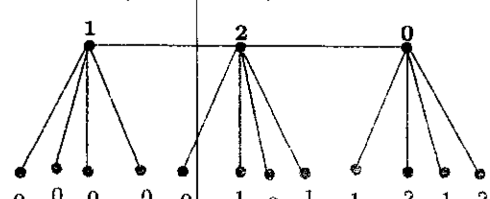

Figure 12. $P_3 \circ 4K_1$

Case 5. $n = 1$ and $n < m$.

When $n = 1$, by assigning the color 0 to u_1 and 1 to all the vertices $v_{11}, v_{12}, \ldots,$ v_{1m}, the result is obtained

Proposition 2.10. *Let G be a graph obtained by duplicating each of the vertex of a cycle C_n by an edge. Then* $\phi(G) = \begin{cases} 5 & \text{for } n \geq 5 \\ n & \text{for } n = 3, 4. \end{cases}$

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of the cycle and $u_{i,1}u_{i,2}$ be the duplicating edge corresponding to the vertex v_i, $0 \leq i \leq n-1$. Assume that $n \geq 5$ in G. Since $\Delta(G) = 4$ and n vertices are of degree 4, $\phi(G) \leq 5$. Color the vertices v_0, v_1, v_2, v_3 and v_{n-1} by $0, 1, 2, 3$ and 4 respectively. Color the vertices $v_4, v_5, \ldots, v_{n-2}$ by 1 and 2 consecutively. The colors other than $c(v_{i-1})$ and $c(v_{i+1})$ (the addition is addition modulo n) are to be assigned to the vertices $u_{i,1}$ and $u_{i,2}$ respectively. Then v_0, v_1, v_2, v_3 and v_{n-1} are the members of the color classes of the colors 0,1,2,3 and 4 respectively in which they are having all the remaining colors as neighbors. Thus $\phi(G) = 5$.

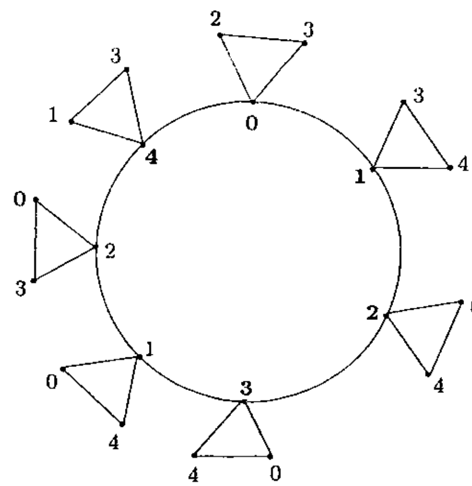

Figure 13

When $n = 4, n$ vertices are of degree 4 and all the remaining are of degree 2. Therefore $\phi(G) \leq n$. Color the vertices v_0, v_1, v_2, v_3 by the colors 0,1,2,3 and the vertices $u_{i,1}$ and $u_{i,2}$ by $c(u_i) + 1, c(u_i) + 2$ (the addition is addition modulo 5) respectively, $0 \leq i \leq 3$. Then $\phi(G) = 4$. When $n = 3$, only 3 vertices are of maximum degree and all the remaining vertices are of degree 2. Therefore $\phi(G) \leq 3$. As in the case of $n = 4$, we may color the vertices of G by 0, 1 and 2 and hence $\phi(G) = 3$.

Proposition 2.11. *Let G be a graph obtained by duplicating each of the vertex of a path P_n by an edge. Then* $\phi(G) = \begin{cases} 5 & \text{if } n \geq 7 \\ 4 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } 1 \leq n \leq 3. \end{cases}$

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of the path and $u_{i,1}u_{i,2}$ be the duplicating edge corresponding to the vertex $v_i, 0 \leq i \leq n-1$. Assume that $n \geq 3$ in G. $n-2$ vertices are having the maximum degree 4. Therefore $\phi(G) \leq 5$. When $n \geq 7$, at least 5 vertices are of degree 4. Assign the colors for the vertices of G as follows:

For $0 \leq i \leq n-1$,

$$C(v_i) = i(mod\, 5)$$
$$C(u_{i,1}) = i + 2(mod\, 5) \text{ and}$$
$$C(u_{i,2}) = i + 3(mod\, 5).$$

Then v_1, v_2, v_3, v_4, v_5 are the members of the color classes of the colors 1,2,3,4 and 0 respectively in which they are adjacent to at least one member of all the remaining color classes. Thus $\phi(G) = 5$. When $4 \leq n \leq 6$, at most 4 vertices of degree 4. So $\phi(G) \leq 4$. By taking congruence modulo 4 in the coloring as in the case of $n \geq 7$, it follows that $\phi(G) = 4$. When $1 \leq n \leq 3$, b-coloring with 4 colors does not exist and a b-coloring with 3 colors is shown in Figure 14.

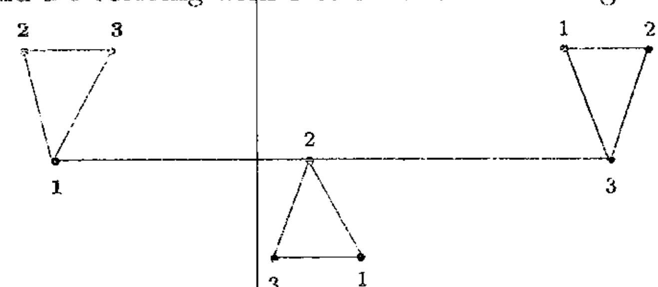

Figure 14

REFERENCES

[1] N. Alon and B. Mohar, The chromatic number of graph power, *Combinatorics Probability and Computing*, **11** (1993), 1–10.

[2] I. W. Irving and D. F. Manlove The b-chromatic number of a graph, *Discrete Appl. Math.*, **91** (1999), 127–141.

[3] J. Kratochvil, Zs. Tuza and M. Voigt, On the b-chromatic number of graphs, *Lecturer Notes in Computer Science*, Springer Berlin, **2573** (2002), 310–320.

[4] M. Kouider, *b-chromatic number of a graph, subgraphs, degrees*, Res. Rep. 1392, LRI, Univ. Orsay, France, 2004.

[5] R. Javedi and B. Omoomi, On b-coloring of Cartesian product of graphs, *Ars Combin.*, **107** (2012), 521–536.

[6] K. Thilagavathi, D. Vijayalakshmi and N. Roopesh, b-coloring of central graphs, *International Journal of Computer Applications*, **3**(11) (2010), 27–29.

[7] M. Venkatachalam, and Vivin. J. Vernold, The b-chromatic number of star graph families, *LE Mathematiche*, **65** (2010), 119–125.

[8] Vivin. J. Vernold and M. Venkatachalam, The b-chromatic number of corona graphs, *Util. Math.*, **88** (2012), 299–307.

[9] D. Vijayalakshmi and K. Thilagavathi, b-coloring in the context of some graph operations, *International Journal of Mathematical Archive*, **3**(4) (2012), 1439–1442.