Regular Elements in a Γ-incline

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Abstract

Necessary and sufficient conditions for a Γ–incline to be regular are obtained. It is proved that every commutative regular Γ–incline is a distributive lattice. Characterizations of the generalized inverses of an element in a Γ–incline are obtained as a generalization and development of regular elements in an incline.

Key words : Incline, Γ–incline, distributive lattice, regular incline

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1. Introduction

The notion of incline and their applications are described comprehensively by Cao, Kim and Roush[2]. Kim and Roush have surveyed and outlined the algebraic properties of incline and of matrices over incline [4]. Inclines are generalization of Boolean Algebra, Fuzzy Algebra and a special type of semiring. Inclines are additively idempotent semirings in which products are less than are equal to either factor. An element a in an incline R is said to be regular if there exists $x \in R$ such that $axa = a$, x is called the g-inverse of a. It is demoted as a- and the set of all g-inverses denoted as $a\{1\}$. An incline R is said to be a regular incline if every element of R is regular [4].

The concept of Γ–ring introduced by Nobusawa [7] as a generalization of ring was later developed by Barnes [1]. Recently Mukherjee has studied about prime ideals, idempotency and commutativity on Γ–rings in [6].

In [3], Chinram and Siammai have discussed about Γ–semigroup S, if a regular element in $\mathcal{D}$–class then every element of $\mathcal{D}$–class are regular and for each $\mathcal{L}$-class and $\mathfrak{R}$-class contains atleast one idempotent if $\mathcal{D}$-class is regular.

In this paper, we introduce the concept of a Γ-incline as an extension of an incline [2] and generalization of Γ–ring [1,6]. We establish some characterization of regular elements in a Γ–incline. In section 2, we present the basic definition and results required

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on inclines and Γ–ring. In section 3, equivalent conditions for regularity of an element in a Γ–incline are obtained as a generalization of regular elements found in our earlier work [5]. For regular elements in a Γ–incline it is proved that equality of right ideals coincide with equality of left ideals.

2. Preliminaries

In this section, we shall present some definitions found in [1,3 and 6].

Definition 2.1

A Γ–ring M is said to be commutative if $a\gamma b = b\gamma a$, for all a, $b \in M$ and $\gamma \in \Gamma$.

Definition 2.2

An element e in a Γ–ring is said to be γ-idempotent (or) simply an idempotent for fixed Γ, if there exists $\gamma \in \Gamma$ such that $e\gamma e = e$.

Definition 2.3

An element a of a Γ–semigroup S is said to be regular if there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha\, x\, \beta a$. A regular Γ–semigroup is a Γ–semigroup in which each element of S is regular.

Definition 2.4

A subset A of the Γ–ring M is a right(left) ideal of M if A is an additive subgroup of M and $A\Gamma M = \{ a\alpha c \,/\, a \in A, \alpha \in \Gamma, c \in M \}$ ($M\Gamma A$) is contained in A. If A is both a left and a right ideal then A is a two-sided ideal, or simply an ideal of M.

3. Regular elements in Γ-incline

In this section, we introduce the concept of a Γ–incline and we derive a set of equivalent conditions for regularity of an element in a Γ-incline. We exhibit that a regular commutative Γ–incline is a distributive lattice. The equality of right (left) ideals of a pair of elements in a regular Γ–incline reduces to equality of the elements.

Definition 3.1

Let M and Γ are additive idempotent abelian semigroup and a mapping $M \times \Gamma \times M \to M$, written as $(a,\alpha,b) \to (a\alpha b)$. Then M is called a Γ-incline if M satisfies the following:

$$x+y = y+x\,,\; x+(y+z) = (x+y)+z\,,\; x\alpha(y+z) = x\alpha y + x\alpha z\,,\; (y+z)\alpha x = y\alpha x + z\alpha x$$

$$x\alpha(y\beta z) = (x\alpha y)\beta z\,,\; x+x = x\,,\; x + x\alpha y = x\,,\; y + x\alpha y = y$$

for all $x,y,z \in M$ and $\alpha, \beta \in \Gamma$.

In particular for M = Γ, it reduces to the definition of an incline [4]. Let (M, $\leq$) is a Γ-incline with order relation defined as $x\alpha y \leq x$ if and only if $x+x\alpha y = x$, simply M denotes the Γ-incline with order relation $\leq$.

Example 3.2

Let M be an arbitrary incline and let Γ be a semigroup . Define a mapping $M \times \Gamma \times M \rightarrow M$ by $a\alpha b = ab$, for all $a,b\in M$ and $\alpha\in\Gamma$. It is easy to see that M is a Γ-incline. Thus an incline can be considered to be a Γ-incline.

Definition 3.3

Let M be a Γ-incline. An element a in M is said to be regular if there exist $x\in M$ and $\alpha,\beta\in\Gamma$ such that $a\alpha x\beta a = a$ and x is called a 1-inverse of a, denoted as a^-.

Remark 3.4

All idempotent elements are regular in a Γ−incline for some Γ.

Definition 3.5

An element $a\in M$ is called anti-regular if there exist an element $x\in M$ and $\alpha,\beta\in\Gamma$, such that $x\beta a\alpha x = x$ and x is called a 2-inverse (or) anti-inverse of a.

Definition 3.6

For $a\in M$, if there exists $x\in M$ and $\alpha,\beta\in\Gamma$, such that $a\alpha x\beta a = a$, $x\beta a\alpha x = x$ and $a\alpha x = x\beta a$, then x is called the Group inverse of a. The Group inverse of a is a commuting 1-2 inverse of a.

Property 3.7

For x, y in a Γ-incline M, $x + y \geq x$ and $x + y \geq y$.
For $x + y = (x + x) + y = x + (x + y)$ and $x + y = x + (y + y) = (x + y) + y$
Thus $x + y \geq x$ and $x + y \geq y$.

Property 3.8 For $x,y\in M$ and $\alpha\in\Gamma$, $x\alpha y \leq x$ and $x\alpha y \leq y$.

Lemma 3.9

Let $a\in M$ be regular. Then $a = a\alpha x = x\beta a$ for all $x\in a\{1\}$ and for some $\alpha,\beta\in\Gamma$.

Proof

If a is regular, then by Property (3.8)
$a = a\alpha x\beta a \leq a\alpha x \leq a$

Therefore $a\alpha x = a$.
Similarly, from $a \leq x\beta a \leq a$, it follows that $a = x\beta a$.
Thus, $a = a\alpha x = x\beta a$ for all $x \in a\{1\}$.

Lemma 3.10

For $a \in M$, a is regular if and only if a is α–idempotent and β–idempotent.

Proof

Let $a \in M$ be regular. Then by Lemma (3.9) $a = a\alpha x = x\beta a$ for all $x \in a\{1\}$.
$a = a\alpha x\beta a = (a\alpha x)\beta a = a\beta a$.
$a = a\alpha x\beta a = a\alpha(x\beta a) = a\alpha a$.
Thus a is α–idempotent and β–idempotent.
Converse is trivial.

Proposition 3.11

If a is regular, then a is the smallest g-inverse of a, that is, $a \leq x$ for all $x \in a\{1\}$.

Proof

Let a be regular, then by Lemma (3.10), $a \in a\{1\}$. By Lemma (3.9) $a = a\alpha x$ for all $x \in a\{1\}$. Hence by Property (3.8) $a \leq x$. Thus a is the smallest g-inverse of a.

Preposition 3.12

Let M be a commutative Γ-incline, M is regular then M is a distributive lattice.

Proof

Let M be a regular Γ-incline then by Lemma (3.10) every element of M is α–idempotent and β–idempotent for some $\alpha, \beta \in \Gamma$. For any $x, y \in M$, $(x + y)\alpha(x + y) = x\alpha x + x\alpha y + y\alpha x + y\alpha y = x + x\alpha y + y\alpha x + y = x + y$ (By Definition (3.1))
Similarly we can prove for $(x + y)\beta(x + y) = x+y$.
$x + y$ is α–idempotent and β–idempotent and hence $x + y$ is a regular element in M
$(x\alpha y)\alpha(x\alpha y) = x\alpha(y\alpha x)\alpha y = (x\alpha x)\alpha(y\alpha y) = x\alpha y$
Similarly we can prove for $(x\alpha y)\beta(x\alpha y) = x\beta y$.
$x\alpha y$ is α–idempotent and β–idempotent and hence $x\alpha y$ is a regular element in M for some $\alpha, \beta \in \Gamma$
By Property (3.7) $x + y \geq x$ and $x + y \geq y$
$x + y$ = least upper bound of $\{x, y\}$ in M
Now, let us take an element $z \in M$, $z \leq x$ and $z \leq y$ then, $z\alpha z \leq x\alpha y \Rightarrow z \leq x\alpha y$.

$x\alpha y$ = greatest lower bound of $\{x, y\}$ in M.
Thus M is a distributive lattice.

Theorem 3.13

For $a \in M$, the following are equivalent:

(i) a is regular
(ii) a is α–idempotent and β–idempotent for some $\alpha,\beta \in \Gamma$
(iii) $a\{1,2\} = \{a\}$
(iv) Group inverse of a exists and coincides with a
(v) $v(\beta a)2 = a = (a\alpha)2$ u for some $u, v \in a\{1\}$ and for some $\alpha,\beta \in \Gamma$.

Proof

(i) ⇒ (ii) This is precisely Lemma (3.10).
To prove the theorem it is enough to prove the following implications:

(ii) ⇒ (iii) ⇒ (iv) ⇒ (i) and (i) ⇒ (v) ⇒ (ii).

(ii)⇒(iii) If a is α–idempotent and β–idempotent, then $a \in a\{1\}$. For any $x \in a$ $\{1, 2\}$ we have $x = x\beta a\alpha x$ and by Lemma (3.9) we get $x = (x\beta a)\alpha x = a\alpha x = a$. Therefore $a\{1, 2\} = \{a\}$

Thus (iii) holds.

(iii) ⇒ (iv) If $a\{1,2\} = \{a\}$ then a is the only commuting 1-2 inverse of a. Therefore by Definition (3.6) the Group inverse of a exists and coincides with a.

(iv) ⇒ (i) This is trivial.

(i)⇒ (v) Let a be regular, then by Lemma (3.9), for some v, $u \in a\{1\}$, $a = (a\alpha v)\beta a = (v\beta a)\beta a = v(\beta a)^2$

Similarly $a = (a\alpha)^2u$

Thus (v) holds.

(v)⇒(ii) Let $a = v(\beta a)^2$ and $a = (a\alpha)^2u$ for some $v, u \in a\{1\}$.
By Property (3.8), $a = v(\beta a)^2 \leq v\beta a \leq a$

$$\Rightarrow a = v\beta a = v(\beta a)^2$$

$$a = v(\beta a)^2 = (v\beta a)\beta a = a\beta a$$

Therefore a is β–idempotent.
In similar manner we have α–idempotent.

Thus (ii) holds.

Hence the proof.

Lemma 3.14

Let M be a regular Γ–incline. For a,b,c$\in$M and $\alpha,\beta\in\Gamma$ the following hold:

(i) $b = y\alpha a \Rightarrow b \Rightarrow b\alpha a \Rightarrow M\Gamma b \subseteq M\Gamma a$

(ii) $c = a\beta x \Rightarrow c \Rightarrow a\beta c \Rightarrow c\Gamma M \subseteq a\Gamma M$.

Proof

(i) Let $b = y\alpha a$, since a is regular. By Lemma (3.10)

$$b\alpha a = y\alpha(a\alpha a) = y\alpha a = b$$

Thus $b = y\alpha a \Rightarrow b\alpha a = b$

Let $b\alpha a = b$ then for $z \in M\,\Gamma b$

$$z = x\,\gamma\, b \text{ for some } x\in M \text{ and } \gamma\in\Gamma$$

$$= (x\,\gamma\, b)\alpha a \in M\,\Gamma a$$

Thus $b = b\alpha a \Rightarrow M\,\Gamma b \subseteq M\,\Gamma a$

Since b is regular, by Lemma (3.9)

$$b = b^{-}\alpha b \in M\,\Gamma b, \text{ since } M\,\Gamma b \subseteq M\,\Gamma a$$

$$b = y\alpha a \text{ for some } y\in M.$$

Thus (i) holds.

(ii) It can be proved along the same lines as of (i) and hence omitted.

Theorem 3.15

For a,b in a regular Γ–incline and $\alpha,\beta\in\Gamma$, we have the following:

$M\Gamma a = M\Gamma b \Leftrightarrow a\Gamma M = b\Gamma M \Leftrightarrow a = b$.

Proof

Since $M\Gamma a = M\Gamma b$, implies $M\,\Gamma a \subseteq M\,\Gamma b$ and $M\,\Gamma b \subseteq M\,\Gamma a$

By Lemma (3.14) (i) we have

$M\,\Gamma a \subseteq M\,\Gamma b \Rightarrow a = a\alpha b \Rightarrow a \leq b$ (By Property (3.8))

and $M\,\Gamma b \subseteq M\,\Gamma a \Rightarrow b = b\beta a \Rightarrow b \leq a$ (By Property (3.8))

Therefore $a = b$. In a similar manner we can show $a\Gamma M = b\Gamma M \Rightarrow a = b$. On the other hand $a = b$ automatically implies $M\Gamma a = M\Gamma b$ and $a\Gamma M = b\Gamma M$.

4. Conclusion

The main results in the present paper are the generalization of the available results shown in the reference for elements in a regular incline [5].

We have introduced the concept of Γ−incline as an extension of an incline. An element a is regular if and only if a is α−idempotent and β−idempotent and a is the only 1−2 inverse of a. For elements in a Γ−incline it is proved that equality of right(left) ideals coincide with equality of elements.

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