Orderings on Generalized Regular Fuzzy Matrices

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Abstract

In this paper, a special type of ordering for k - regular fuzzy matrices is introduced as a generalization of the minus partial ordering for regular fuzzy matrices. A set of equivalent conditions for a pair of k – regular matrices to be under this ordering are obtained. We exhibit that this ordering is preserved under similarity relation.

Key words : Fuzzy matrix, Regular, k – regular, Generalized inverse.

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1. Introduction

We deal with fuzzy matrices that is, matrices over the fuzzy algebra $\mathcal{F}$ with support [0,1] and fuzzy operations {+ , . } defined as a+b = max{a, b }, a . b = min {a,b} for all a , b $\in \mathcal{F}$. Let $\mathcal{F}_{m,n}$ be the set of all mxn fuzzy matrices over $\mathcal{F}$. In short $\mathcal{F}_n$ denotes $\mathcal{F}_{nn}$. The row space R(A) is the subspace of $\mathcal{F}_{1,n}$ generated by the rows of A, the column space C(A) is defined in the dual fashion. A matrix A$\in \mathcal{F}_{mn}$ is said to be regular if there exists X$\in \mathcal{F}_{nm}$ such that A X A = A, X is called a generalized inverse (ginverse) of A. Let A{1} denotes the set of all g-inverses of A.

A Matrix A$\in \mathcal{F}_n$ is said to be right(left) k- regular if there exists X (Y) $\in \mathcal{F}_n$, such that $A^k X A = A^k$ ($AYA^k = A^k$), X(Y) is called a right (left) k-g inverse of A, where k is a positive integer.. Let $A_r\{1^k\}$ and $A_l(1^k)$ be the set of all right k-g inverses and left k-g inverses of A$\in F_n$. In [3] it has been proved that right k-g inverse and left k-g inverse for a fuzzy matrix are distinct. In this paper, by a k-regular matrix, we mean that it is either right or left k-regular. If A is k-regular, then it is h regular for all h $\geq$ k. Let $\mathcal{F}_n^{(k)}$ denotes the set of all k-regular fuzzy matrices. Let $A\{1^k\} = A_r\{1^k\} \cup A_l(1^k)$ be the set of all k-g inverse of A. In particular, for k=1, it reduces to a regular matrix and set of all its g-inverses. If A is k – potent, that is, k is the smallest positive integer such that $A^k = A$, then k regularity coincides with regularity. If A^k is regular then A is k-regular, for if A^k is regular, then there exists X$\in \mathcal{F}_n$ such that $A^kXA^k = A^k$ and hence $A^kYA = A^k = AZA^k$ for Y = XA^k-1 and Z=A^k-1 X. In this paper, we introduce a special type of ordering for k-regular fuzzy matrices as a generalization of the minus ordering studied in [2] for regular fuzzy matrices.

In the sequel, we shall make use of the following results found in [1].

Lemma 1.1

For $A, B \in \mathcal{F}_n$,

$R(A) \subseteq R(B) \Leftrightarrow A = XB$ for some $X \in \mathcal{F}_n$

$C(A) \subseteq C(B) \Leftrightarrow A = BY$ for some $Y \in \mathcal{F}_n$.

Lemma 1.2

For $A \in \mathcal{F}_n^{(k)}$ and $X \in \mathcal{F}_n$, $X \in A_r\{1^k\} \Leftrightarrow X^T \in A_l^T(1^k)$

2. Orderings on k–Regular Fuzzy Matrices

In this section, we define a special type of ordering and called it as k – ordering for k-regular fuzzy matrices. Some basic properties on a pair of k – regular fuzzy matrices under this ordering are discussed.

Definition 2.1

For $A \in \mathcal{F}_n^{(k)}$, $B \in \mathcal{F}_n$; the ordering denoted as $A \,\overline{<}\, B$ is defined as

$$A \,\overline{<}\, B \Leftrightarrow \quad A^k X = B^k X \text{ for some } X \in A_r\{1^k\}$$
$$\text{and} \quad YA^k = YB^k \text{ for some } Y \in A_l\{1^k\}$$

In particular for k = 1, Definition (2.1) reduces to minus ordering on fuzzy matrices found in [2]. If $X \in A_r\{1^k\}$, then X need not be a g – inverse of A^k. This is illustrated in the following example.

Example 2.1

Let $A = \begin{pmatrix} 0.3 & 0.7 \\ 0.5 & 0 \end{pmatrix}$, $\quad A^2 = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}$

For $X = \begin{pmatrix} 0.3 & 0.7 \\ 0.5 & 0 \end{pmatrix}$, $A^2XA = A^2$. Hence $X \in A_r\{1^2\}$ and A is 2 – regular, but $A^2XA^2 \neq A^2$, therefore X is not in $A^2\{1\}$. Thus X is a 2 – g inverse of A but X is not a g – inverse for A^2.

Lemma 2.1

For $A \in \mathcal{F}_n^{(k)}$ and $B \in \mathcal{F}_n$; the following are equivalent.

(i) $A \,\overline{<}\, B$

(ii) $A^k = B^k X A = AYB^k$ for some $X, Y \in A\{1^k\}$

Proof

(i) $\Rightarrow$ (ii)

$A \bar{<} B \Rightarrow A^k X = B^k X$ for some $X \in A_r\{1^k\}$
and $Y A^k = YB^k$ for some $Y \in A_l\{1^k\}$

Now, $A^k = (A^kX) A = B^k X A$ for some $X \in A_r\{1^k\}$

$A^k = A (YA^k) = AY B^k$ for some $Y \in A l\{1^k\}$

$A^k = B^k X A = AY B^k$ for some $X, Y \in A\{1^k\}$

Thus (ii) holds.

(ii) $\Rightarrow$ (i)

Let $Z = XAX$ for $X \in A_r\{1^k\}$

$A^kZA = A^k (XAX) A = (A^kX A) XA = A^k X A = A^k$

$\Rightarrow Z \in A_r\{1^k\}$

Similarly, $AZA^k = A^k$ for $Z = YAY$ for $Y \in A_l\{1^k\}$

$\Rightarrow Z \in A_l\{1^k\}$

Thus, for $X \in A\{1^k\}$, $Z = XAX \in A_r\{1^k\}$ when $X \in A_r\{1^k\}$ and $Z = XAX \in A_l\{1^k\}$ when $X \in A_l\{1^k\}$.

Now, $A^kZ = A^k (XAX) = (A^k XA)X = A^k X = (B^k XA) X = B^k (XAX) = B^kZ$

Hence $A^kZ = B^kZ$ for some $Z \in A_r\{1^k\}$.

Similarly, $ZA^k = ZB^k$ for some $Z \in A_l\{1^k\}$.

Therefore $A \bar{<} B$. Thus (i) holds.

Hence the Theorem.

Lemma 2.2

For $A, B \in \mathcal{F}_n^{(k)}$

(i) If B is right k-regular and $R(A^k) \subseteq R(B^k)$ then $A^k = A^kB^-B$ for each right k-g inverse B^- of B.

(ii) If B is left k-regular and $C(A^k) \subseteq C(Bk)$ then $A^k = BB^-A^k$ for each left k-g inverse B^- of B.

Proof

(i) $R(A^k) \subseteq R(B^k) \Rightarrow A^k$	$= XB^k$	(By Lemma 1.1)
	$= XB^kB^-B$	(for each $B^- \in B_r\{1^k\}$)
	$= A^kB^-B$	

Thus (i) holds.

(ii) $C(A^k) \subseteq C(B^k) \Rightarrow A^k$	$= B^kY$	(By Lemma 1.1)
	$= BB^-B^kY$	(for each $B^- \in B_l\{1^k\}$)
	$= BB^-A^k$	

Thus (ii) holds.

Theorem 2.1

For $A,B \in \mathfrak{F}_n^{(k)}$, if $A \bar{<} B$ then $R(A^k) \subseteq R(B^k)$, $C(A^k) \subseteq C(B^k)$ and $A^kX\,B = A^k = BYA^k$ for each $X \in Br\{1^k\}$ and for each $Y \in B_l\{1^k\}$.

Proof

$A \bar{<} B \quad \Rightarrow A^k = AY\,B^k = B^k\,X\,A$ (By Lemma 2.1)

$\Rightarrow A^k = VB^k = B^k\,U$, where $V = AY$ and $U = XA$

$\Rightarrow R(A^k) \subseteq R(B^k)$ and $C(A^k) \subseteq C(B^k)$ (By Lemma 1.1)

$A^kXB = A^k = BYA^k$ for each $X \in B_r\{1^k\}$ and for each $Y \in B_l\{1^k\}$ (By Lemma 2.2)

Theorem 2.2

For $A,B \in \mathfrak{F}_n^{(k)}$, the following hold.

(i) $A \bar{<} A$

(ii) $A \bar{<} B$ and $B \bar{<} A$ then $A^k = B^k$

(iii) $A \bar{<} B$ and $B \bar{<} C$ then $A \bar{<} C$

Proof

(i) $A \bar{<} A$ is trivial.

(ii) $A \bar{<} B \Rightarrow A^k = B^k\,XA$ for $X \in A_r\{1^k\}$ (By Lemma 2.1)

$B \bar{<} A \Rightarrow B^k = BY\,A^k$ for $Y \in A_l\{1^k\}$ (By Lemma 2.1)

Now, $A^k = B^k\,XA = (BYA^k)XA = BY(A^k\,XA) = BYA^k = B^k$

Hence $A \bar{<} B$ and $B \bar{<} A \Rightarrow A^k = B^k$

(iii) $A \bar{<} B \Rightarrow A^k = A^k\,B^-B = BB^-A^k$ (By Theorem 2.1 and Lemma 2.2(i))

$B \bar{<} A \Rightarrow B^k = C^k\,B^-B = BB^-C^k$ (By Theorem 2.1 and Lemma 2.2(i))

Let $Z = B^-BX$ for $B^- \in B_r\{1^k\}$ and $X \in A_r\{1^k\}$

Then $A^k\,ZA = (A^kB^-B)XA = A^k\,XA = A^k$

Therefore $Z \in A_r\{1^k\}$

If $Z = YBB^-$ for $B^- \in B_l\{1^k\}$ and $Y \in A_l\{1^k\}$ then it follows that $AZA^k = A^k$

Therefore $Z \in A_l\{1^k\}$.

Since $A \bar{<} B$ and $B \bar{<} C$, applying Theorem 2.1, we have

$A^k\,Z = A^k\,(B^-BX)$

$= (\,A^k\,B^-B)X$

$= A^k\,X$ (By Theorem 2.1)

$= B^k\,X$

$= (C^k\,B^-B)X$ (By Lemma 2.1)

$= (BB^-B^k)x$

$= C^k\,(B^-B\,X)$

$= C^k\,Z$ for some $Z \in A_r\{1^k\}$.

and $ZA^k = Z\,C^k$ for some $Z \in A_l\{1^k\}$ can be proved in a similar manner.
Hence $Z \in A\{1^k\}$ with $A^k Z = C^k Z$ and $ZA^k = Z\,C^k$. Therefore $A \,\overline{<}\, C$.

Remark 2.1

In particular for k = 1, Theorem(2.2) reduces to Theorem(2.2) of [2], that is, the minus ordering is a partial ordering on regular matrices.

3. Properties of k-ordering

In this section, we shall derive some basic properties of k – ordering on k – regular fuzzy matrices that include the results found in [2] as a special case.

Proposition 3.1

For $A,B \in \mathfrak{F}_n^{(k)}$, $A \,\overline{<}\, B \Leftrightarrow A^T \,\overline{<}\, B^T$

Proof

$A \,\overline{<}\, B \Leftrightarrow \quad A^k A^- = B^k A^-$ for some $A^- \in A_r\{1^k\}$

and $\quad A^- A^k = A^- B^k$ for some $A^- \in A_l\{1^k\}$

By Lemma (1.2), $A^- \in A_r\{1^k\} \Leftrightarrow (A^-)^T \in A_l\{1^k\}$.

$$A^k A^- = B^k A^-$$
$$\Leftrightarrow (A^k A^-)^T = (B^k A^-)^T$$
$$\Leftrightarrow (A^-)^T (A^k)^T = (A^-)^T (B^k)^T$$
$$\Leftrightarrow (A^T)^- (A^k)^T = (A^T)^- (B^k)^T$$

Thus $A^k A^- = B^k A^- \Leftrightarrow (A^T)^- (A^k)^T = (A^T)^- (B^k)^T$

Similarly $A^- A^k = A^- B^k \Leftrightarrow (A^k)^T (A^T)^- = (B^k)^T (A^T)^-$

Hence $A \,\overline{<}\, B \Leftrightarrow AT \,\overline{<}\, B^T$

Proposition 3.2

For $A,B \in \mathfrak{F}_n^{(k)}$, $A \,\overline{<}\, B \Leftrightarrow PAP^T \,\overline{<}\, PBP^T$ for some permutation matrix P.

Proof

Since A is k- regular, it can be verified that PAP^T is k- regular and $PA^- P^T$ is a k-g inverse of PAP^T for each k-g inverse A^- of A.

$$\begin{aligned}
\text{Now, } (PAP^T)^- (PAP^T)^k &= PA^- P^T PA^k P^T \\
&= PA^-(P^T P) A^k P^T \\
&= P(A^- A^k)P^T \\
&= P(A^- B^k)P^T \\
&= (PA^- P^T)(PB^k P^T) \\
&= (PAP^T)^- (PBP^T)^k
\end{aligned}$$

$$\text{Hence } (PAP^T)^- (PAP^T)^k = (PAP^T)^- (PBP^T)^k$$

$$\text{Similarly } (PAP^T)^k (PAP^T)^- = (PBP^T)^k (PAP^T)^-$$

Hence $(PAP^T) \overline{<} (PBP^T)$

Conversely, if $PAP^T \overline{<} PBP^T$, then by the preceding part,

$$A = P^T(PAP^T)P \overline{<} P^T(PBP^T)P = B$$

Thus $A \overline{<} B$.

Proposition 3.3

For $A, B \in \mathfrak{F}_n^{(k)}$, if $A \overline{<} B$ with B^k is idempotent, then A^k is idempotent.

Proof

Since $A \overline{<} B$, By Lemma (2.1)

$$\begin{aligned} A^{2k} &= A^k A^k \\ &= (AYB^k)(B^kXA) \\ &= AY(B^{2k})XA = (AYB^k)XA = A^k X A = A^k \end{aligned}$$

Remark 3.1

In the above Proposition 3.3, if $A \overline{<} B$ with A^2 idempotent then B^2 need not be idempotent. This is illustrated in the following.

Example 3.1

Consider $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ Here $A \overline{<} B$ for $A^-=A$, But B is not idempotent.

Proposition 3.4

For $A, B \in \mathfrak{F}_n^{(k)}$, if $A \overline{<} B$ then $B^k = 0$ implies $A^k = 0$.

Proof

Since $A \overline{<} B \Rightarrow A^k = AYB^k$ (By Lemma 2.1)

$= 0$

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