M/M/1 Retrial Queueing System with Single Working Vacation under Non-Pre-Emptive Priority Service

A. Muthu Ganapathi Subramanian
Tagore Arts College, Puducherry – 605 008, India
E-mail: csamgs1964@gmail.com

G. Ayyappan
Pondicherry Engineering College, Puducherry – 605 008, India
E-mail: ayyappanpec@gmail.com

Gopal Sekar
Tagore Arts College, Puducherry – 605 008, India
E-mail: gopsek28@yahoo.co.in

Abstract

Consider a single server retrial queueing system with non-pre-emptive priority service and single working vacation in which two types of customers arrive in a Poisson process with arrival rates λ_1 for low and λ_2 for high priority customers. We assume that regular service times follow an exponential distribution with parameters μ_1 and μ_2 correspondingly. The retrial is introduced for low priority customers only. During working vacation the server's service to the arriving customers with lesser service rates μ_3 and μ_4 correspondingly which also follow the exponential distribution. However the server may return from the working vacation with a working vacation rate α which follows an exponential distribution. The access from orbit to the service facility follows the classical retrial policy and the high priority customers will be governed by the non-pre-emptive priority service. This model is solved by using Matrix geometric Technique.

Keywords: Single Server – Stochastic nature – priority –non-pre-emptive priority service – working vacation - Matrix Geometric Method – Orbit – classical retrial policy

Mathematics subject classification: **60K25, 65K30.**

1. Introduction

Queueing systems in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time as discussed by Artalejo [1] in his bibliography, is called Retrial queues. Because of the complexity of the retrial queueing models, analytic results are generally difficult to obtain. There are a great number of numerical and approximations methods are available, in this paper we will place more emphasis on the solutions by Matrix geometric method discussed by Gomez [4].

2. Description of the Queueing System

Consider a single server retrial queueing system with single working vacation under **non-pre-emptive priority service** [2,3,5] in which two types of customers arrive in a Poisson process with arrival rate λ_1 for low priority customers and λ_2 for high priority customers. These customers are identified as primary calls. In this model the server provides two types of service namely Regular service and Lesser service. The regular service times follow an exponential distribution with parameters μ_1 and μ_2 for low and and high priority customers respectively.The lesser service times during the working vacation follows an exponential distribution with parameter μ_3 and μ_4 for low and high priority customers respectively. The working vacation time follows an exponential distribution with parameter α. The retrial is introduced for low priority customers only. Let k be the maximum number of waiting spaces for high priority customers in front of the service station.

2.1 Description of the Working Vacation

Working vacation models studied by Liu [6], Tian [8], Tien Van Do [9], Wu [10], is a kind of semi-vacation policy and it was first introduced by Servi and Finn [7]. A customer is served at a lesser service rate rather than completely stopping the service during a vacation. Part of service ability keeps the system operating in a lesser speed during a vacation. In the classical vacation queueing models, the server completely stops the service, but under working vacation policy, the server can still work during the vacation.So the working vacation is more reasonable than the classical vacation in some cases. If service speed degenerates into zero in a working vacation, the working vacation queueing model becomes a classical vacation queueing model. Therefore, the working vacation model is the generalization of the classical vacation model. The working vacation period is an operation period with a lower speed. At a vacation completion instant, if there are customers in the system, the server will come back to the normal working level. Otherwise, the server stays in an idle period. Once customers arrive into the system the server immediately begins a new busy period. After completion of a service (low/high), the server has to go for compulsory working vacation provided all the conditions below are satisfied.

1. There are no customers in the service station,
2. There are no customers in the high priority queue and
3. There are no customers (low priority) in the orbit

This is called a single working vacation policy (Exhaustive service type). The server may return from the working vacation is independent of the number of customers in the system. The term single working vacation means the server goes for another working vacation again after completing atleast one service. Assume that the service time's μ_3, μ_4 during the working vacation are lesser than μ_1, μ_2 respectively.

If the server is free at the time of a primary call (low/high), the arriving call begins to be served immediately by the server and customer leaves the system after service completion. Otherwise, if the server is busy then the low priority arriving customer goes to orbit and becomes a source of repeated calls. The pool of source of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity σ. If an incoming repeated call (low) finds the server free, it is served and leaves the system after service, while the source which produced this repeated call disappears. If any one of the waiting spaces is occupied by the high priority customers then the low priority customers (as a primary call) cannot enter into the service station and goes to the orbit. If the server is busy and there are some waiting spaces then a high priority customer can enter into the service station and waits for his service. If there are no waiting spaces then the high priority customers cannot enter into the service station and will be lost for the system. Otherwise, the system state does not change. If the server is engaging with low priority customer and at that time the higher priority customer comes then the high priority customer will get service only after completion of the service of low priority customer who is in service. This type of priority service is called the Non-pre-emptive priority service.

2.2 Retrial Policy

Most of the queueing system with repeated attempts assume that each customer in the retrial group seeks service independently of each other after a random time exponentially distributed with rate σ so that the probability of repeated attempt during the interval $(t, t+\Delta t)$ given that there were n customers in orbit at time t is $n\sigma\,\Delta t + O(\Delta t)$. This discipline for access for the server from the retrial group is called classical retrial rate policy. The input flow of primary calls (low and high), interval between repetitions and service times are mutually independent.

3. Matrix Geometric Solutions

Let N(t) be the random variable which represents the number of low priority customers in the orbit at time t and P(t) be the random variable which represents the number of high priority customers in the queue (in front of the service station) at time t and S(t) represents the server state at time t and C(t) represents working vacation period of system at time t.

The random process is described as

$\{< N(t), P(t), S(t), C(t)> / N(t) = 0,1,2,3,\ldots ; P(t) = 0,1,2,3,\ldots,k ; S(t) = 0,1,2; C(t) = 0,1 \}$.

S(t) = 0 if the server is idle at time t

S(t) = 1 if the server busy with low priority customer at time t

S(t) = 2 if the server busy with high priority customer at time t

C(t) = 0 if the server is in working vacation period at time t

C(t) = 1 if the server is regular period at time t.

The possible state space is

$\{(u,v,w,z) : u = 0,1,2,3,\ldots ;\ v = 0;\ w = 0,1,2; z = 0\} \cup$

$\{(u,v,w,z) : u = 0,1,2,3,\ldots ;\ v = 1,2,3,\ldots,k;\ w = 1,2;\ z = 0\} \cup$

$\{(u,v,w,z) : u = 0,1,2,3,\ldots ;\ v = 0;\ w = 0,1,2;\ z = 1\} \cup$

$\{(u,v,w,z) : u = 0,1,2,3,\ldots ;\ v = 1,2,3,\ldots,k;\ w = 1,2;\ z = 1\}$

The infinitesimal generator matrix **Q** is given below

$$\mathbf{Q} = \begin{pmatrix} A_{00} & A_0 & O & O & O & \cdots \\ A_{10} & A_{11} & A_0 & O & O & \cdots \\ O & A_{21} & A_{22} & A_0 & O & \cdots \\ O & O & A_{32} & A_{33} & A_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Notations

$T_1 = -(\lambda_1+\lambda_2+\alpha)$ $T_2 = -(\lambda_1+\lambda_2+\mu_3+\alpha)$ $T_3 = -(\lambda_1+\lambda_2+\mu_4+\alpha)$ $T_4 = -(\lambda_1+\mu_3+\alpha)$

$T_5 = -(\lambda_1+\mu_4+\alpha)$ $T_6 = -(n\sigma+\lambda_1+\lambda_2+\alpha)$ $T_7 = -(M\sigma+\lambda_1+\lambda_2+\alpha)$ $T_8 = -(\lambda_2+\mu_3+\alpha)$

$T_9 = -(\lambda_2+\mu_4+\alpha)$ $T_{10} = -(\mu_3+\alpha)$ $T_{11} = -(\mu_4+\alpha)$ $S_1 = -(\lambda_1+\lambda_2)$ $S_2 = -(\lambda_1+\lambda_2+\mu_1)$

$S_3 = -(\lambda_1+\lambda_2+\mu_2)$ $S_4 = -(\lambda_1+\mu_1)$ $S_5 = -(\lambda_1+\mu_2)$ $S_6 = -(n\sigma+\lambda_1+\lambda_2)$ $S_7 = -(M\sigma+\lambda_1+\lambda_2)$

$S_8 = -(\lambda_2+\mu_1)$ $S_9 = -(\lambda_2+\mu_2)$ $S_{10} = -(\mu_1)$ $S_{11} = -(\mu_2)$

$\mathbf{A_{00}}, \mathbf{A_{nn-1}}, \mathbf{A_{nn}}, \mathbf{A_{nn+1}}$ are square matrices order (4k+6) for n = 1,2,3. . .

$$\mathbf{A_{00}} = \begin{pmatrix}
T_1 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_3 & T_2 & 0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_4 & 0 & T_3 & 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \mu_3 & T_2 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \mu_4 & 0 & T_3 & 0 & \lambda_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_3 & T_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & T_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & T_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & S_1 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & S_2 & 0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & S_3 & 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \mu_1 & S_2 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_3 & 0 & \lambda_2 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & S_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_3 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & S_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & S_5
\end{pmatrix}$$

$$\mathbf{A}_{n,n-1} = (\mathbf{a}_{ij}) \quad \text{where } a_{ij} \begin{array}{ll} = n\sigma & \text{if } i = 1 \text{ and } j = 2 \\ = 0 & \text{otherwise} \end{array}$$

$$\mathbf{A}_{n,n} = \begin{pmatrix}
T_6 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_3 & T_2 & 0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_4 & 0 & T_3 & 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \mu_3 & T_2 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \mu_4 & 0 & T_3 & 0 & \lambda_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_3 & T_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & T_3 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & T_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & S_6 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mu_1 & S_2 & 0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mu_2 & 0 & S_3 & 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \mu_1 & S_2 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_3 & 0 & \lambda_2 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & S_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_3 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & S_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & S_5
\end{pmatrix}$$

$$\mathbf{A}_{n,n+1} = (\mathbf{a}_{ij}) \quad \text{for } n = 0,1,2,3,\ldots$$

$$\text{where } \mathbf{a}_{ij} \begin{array}{ll} = 0 & \text{if } i = 1 \text{ and } j = 1,2,3,\ldots,4k+6 \\ = \lambda_1 & \text{if } i = j \text{ and } i = 2,3,4,\ldots,4k+6 \\ = 0 & \text{otherwise.} \end{array}$$

If the capacity of the orbit is finite say M then

$$\mathbf{A}_{M,M} = \begin{pmatrix}
T_7 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_3 & T_8 & 0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\mu_4 & 0 & T_9 & 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \mu_3 & T_8 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \mu_4 & 0 & T_9 & 0 & \lambda_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_3 & T_8 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_4 & 0 & T_9 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & T_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & T_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & S_7 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & S_8 & 0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & S_9 & 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \mu_1 & S_8 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_9 & 0 & \lambda_2 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & S_8 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_9 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & S_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & S_{11}
\end{pmatrix}$$

Let **X** be a steady-state probability vector of Q and partitioned as **X**= (x(0),x(1),x(2), ….) and **X** satisfies

$$\mathbf{XQ} = \mathbf{0}, \quad \mathbf{Xe} = \mathbf{1} \tag{1}$$

where $x(i) = (P_{i000}, P_{i010}, P_{i020}, P_{i110}, P_{i120}, P_{i210}, P_{i220}, \ldots, P_{ik10}, P_{ik20}, P_{i001}, P_{i011}, P_{i021}, P_{i111}, P_{i121}, P_{i211}, P_{i221}, \ldots, P_{ik11}, P_{ik21})$ for $i = 0,1,2,3,\ldots$

4. Direct Truncation Method

In this method one can truncate the system of equations in **(1)** for sufficiently large value of the number of customers in the orbit, say M. That is, the orbit size is restricted to M such that any arriving customer finding the orbit full is considered lost. The value of M can be chosen so that the loss probability is small. Due to the intrinsic nature of the system in **(1)** the only choice available for studying M is through algorithmic methods. While a number of approaches are available for determining the cut-off point, M , The one that seems to perform well (w.r.t approximating the system performance measures) is to increase M until the largest individual change in the elements of **X** for successive values is less than ε a predetermined infinitesimal value.

5. Analysis of steady state probabilities

We are applying Direct Truncation Method to find Steady state probability vector **X**. Let M denote the cut-off point or Truncation level. The steady state probability vector $\mathbf{X}^{(M)}$ is now partitioned as $\mathbf{X}^{(M)} = (x(0), x(1), x(2), \ldots x(M))$ and $\mathbf{X}^{(M)}$ satisfies

$$\mathbf{X}^{(M)} \mathbf{Q}^{(M)} = \mathbf{0}, \quad \mathbf{X}^{(M)} \mathbf{e} = \mathbf{1}$$

where $x(i) = (P_{i000}, P_{i010}, P_{i020}, P_{i110}, P_{i120}, P_{i210}, P_{i220}, \ldots P_{ik10}, P_{ik20}, P_{i001}, P_{i011}, P_{i021}, P_{i111}, P_{i121}, P_{i211}, P_{i221}, \ldots, P_{ik11}, P_{ik21})$ $i = 0, 1, 2, 3 \ldots M$

The above system of equations is solved by Numerical method . Since there is no clear cut choice for M, we may start the iterative process by taking, say M = 1 and increase it until the individual elements of **X** do not change significantly. That is, if M^* denotes the truncation point then

$$\| \mathbf{X}^{M^*}(\mathbf{i}) - \mathbf{X}^{M^*-1}(\mathbf{i}) \|_\infty < \varepsilon, \quad \varepsilon \text{ is an infinitesimal quantity.}$$

6. Stability condition

Theorem 6.1

The inequality $\frac{\lambda_1}{\mu_1} < \frac{[(1-x)(1-\chi_{2k}) + x\chi_{2k+1}]}{1-t^k}$ where $x = \lambda_2/\mu_2$, $y = \mu_1/\mu_2$ and $t = x/(x+y)$ is the necessary and sufficient condition for the system to be stable.

Proof

Let Q be an infinitesimal generator matrix for the queueing system (without retrial)

The stationary probability vector X statisfying

$$\mathbf{XQ = 0} \quad \text{and} \quad \mathbf{Xe=1} \tag{2}$$

$$\mathbf{A_0+RA_1+R^2A_2 =0}\ , \ \mathbf{R} \text{ is the Rate Matrix} \tag{3}$$

The system is stable if $sp(R) < 1$

R satisfies $sp(R) < 1$ if and only if $\mathbf{\Pi A_0 e < \Pi A_2 e}$ (4)

where Π is given by $(\pi_0, \pi_1, \pi_2, \ldots, \pi_{2k}, \pi_{2k+1}, \chi_0, \chi_1, \chi_2, \ldots, \chi_{2k}, \chi_{2k+1})$

$$\mathbf{\Pi A = 0 \text{ and } \Pi e =1} \tag{5}$$

$$\mathbf{A=A_0+A_1+A_2} \tag{6}$$

A_0, A_1, A_2 are square matrices of order 4k+4 and

$A_0 = \lambda_1 I$ where I an identity matrix

$$A_1 = \begin{pmatrix}
T_2 & 0 & \lambda_2 & 0 & 0 & 0 & \ldots & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & T_3 & 0 & \lambda_2 & 0 & 0 & \ldots & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \mu_3 & T_2 & 0 & \lambda_2 & 0 & \ldots & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \mu_4 & 0 & T_3 & 0 & \lambda_2 & \ldots & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \mu_3 & T_2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \mu_4 & 0 & T_3 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & T_5 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & S_2 & 0 & \lambda_2 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & S_3 & 0 & \lambda_2 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \mu_1 & S_2 & 0 & \lambda_2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \mu_2 & 0 & S_3 & 0 & \lambda_2 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \mu_1 & S_2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & S_3 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & S_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & S_5
\end{pmatrix}$$

$A_2 = (a_{ij})$ where $a_{ij} = \mu_3$ if $i = 1$ and $j = 1$

$a_{ij} = \mu_4$ if $i = 2$ and $j = 1$

$a_{ij} = 0$ otherwise

By substituting A_0, A_1, A_2 in equation (5) and (6)

$$- (\lambda_2+\alpha)\, \pi_0+\mu_4\pi_1 = 0$$

$$- (\lambda_2+\mu_4+\alpha)\, \pi_1 + \mu_3\pi_2+\mu_4\pi_3 = 0$$

$$\lambda_2 \pi_{2i-2} - (\lambda_2+\mu_3+\alpha)\, \pi_{2i} = 0 \quad (i =1,2,3...k-1)$$

$$\lambda_2 \pi_{2i-1} - (\lambda_2+\mu_4+\alpha)\pi_{2i+1} + \mu_3\pi_{2i+2}+\mu_4\pi_{2i+3} = 0 \quad (i =1,2,3,\ldots,k-1)$$

$$\lambda_2 \pi_{2k-2} - (\mu_3+\alpha)\, \pi_{2k} = 0$$

$$\lambda_2 \pi_{2k-1} - (\mu_4+\alpha)\, \pi_{2k+1} = 0$$

$$\alpha\pi_0 - (\lambda_2+\alpha)\, \chi_0+\mu_2\chi_1 = 0$$

$$\alpha\pi_1 - (\lambda_2+\mu_4+\alpha)\, \chi_1 + \mu_1\chi_2+\mu_2\chi_3 = 0$$

$$\alpha\, \pi_2+ \lambda_2\chi_{2i-2} - (\lambda_2+\mu_1+\alpha)\, \chi_{2i} = 0 \quad (i = 1,2,3,\ldots,k-1)$$

$$\alpha\, \pi_3+ \lambda_2\, \chi_{2i-1} - (\lambda_2+\mu_2+\alpha)\, \chi_{2i+1} + \mu_1\chi_{2i+2}+\mu_2\chi_{2i+3} = 0 \quad (i = 1,2,3,\ldots,k-1)$$

$$\alpha\pi_{2k} +\lambda_2\chi_{2k-2} - (\mu_1+\alpha)\, \chi_{2k} = 0$$

$$\alpha\pi_{2k+1} +\lambda_2\, \chi_{2k-1} - (\mu_2+\alpha)\, \chi_{2k+1} = 0$$

After Simplification of the above equations, we get

$$\alpha\, (\pi_0+\pi_1+ \ldots+\pi_{2k+1}) = 0$$

Therefore, $\pi_0 = \pi_1 = \ldots = \pi_{2k+1} = 0$ since $\alpha \neq 0$, further we get,

$$\chi_1 = x\chi_0$$

$$\chi_{2i} = t^i\, \chi_0\,, \; i=1,2,3,\ldots,k-1$$

$$\chi_{2i+1} = x(\chi_{2i-1}+\chi_{2i})\,, \; i=1,2,3\ldots k-1$$

$$\chi_{2k-1} = x(\chi_{2k-3}+\chi_{2k-2})$$

$$\chi_{2k} = (x/y)\, t^{k-1}\chi_0, \quad \chi_{2k+1} = x\chi_{2k-1}$$

From (5)

$$\boldsymbol{\pi_0+\pi_1+ \pi_2+\pi_3+ \pi_4+\pi_5+\ldots+ \pi_{2k-1}+\pi_{2k}+\pi_{2k+1}+ \chi_0+\chi_1+ \chi_2+\chi_3+ \chi_4+\chi_5+\ldots+ \chi_{2k-1}+ \chi_{2k}+\chi_{2k+1} =1}$$

by substituting values of π_i and χ_i in the above equation we get

$$\chi_0 = \frac{(1-t)[(1-x)(1-\chi_{2k}) + x\chi_{2k+1}]}{1-t^k}$$

From (3), $(\lambda_1/\mu_1) < \chi_0 (1+ x/y)$

by substituting χ_0 we get

$$\frac{\lambda_1}{\mu_1} < \frac{[(1-x)(1-\chi_{2k}) + x\chi_{2k+1}]}{1-t^k} \qquad (7)$$

The inequality (7) is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after departure n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic if (7) satisfies and it is readily to see that $\{Q_n, n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Qn, n \geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n \geq 0\}$ is positive recurrent if $|\psi_k| < \infty$ for all k and $\lim_{k\to\infty} \sup \psi_k < 0$ where

$$\psi_k = E(Q_{n+1} - Q_n / Q_n = k) \quad , k = 0, 1, 2, 3, \ldots$$

$$\psi_k = \left(\frac{\frac{\lambda_1}{\mu_1}(1-t^k)}{(1-x)(1-\chi_{2k}) + x\chi_{2k+1}} - \frac{k\sigma}{\lambda_1 + \lambda_2 + k\sigma} \right)$$

if $\left(\frac{\frac{\lambda_1}{\mu_1}(1-t^k)}{(1-x)(1-\chi_{2k}) + x\chi_{2k+1}} < 1 \right)$, then $|\psi_k| < \infty$ for all k and $\lim_{k\to\infty} \sup \psi_k < 0$.

Therefore the embedded Markov chain $\{Q_n, n>0\}$ is ergodic. If $k \to \infty$ then $\chi_{2k} \to 0$ and $\chi_{2k+1} \to 0$ and $t \to 0$. So the above stability condition becomes $(\lambda_1/\mu_1 + \lambda_2/\mu_2) < 1$.

7. Special Cases

1. This model becomes single server Retrial queueing system with non-pre-emptive priority service if $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$
2. This model becomes single server Retrial queueing system with non-pre-emptive priority service if $\alpha \to \infty$.

3. This model becomes Single server Retrial queueing system with exhaustive type classical vacation under non-pre-emptive priority service as studied by Ayyappan *et al.*,[5] if $\mu_3 \to 0$ and $\mu_4 \to 0$.

8. System Performance Measures

We can find various probabilities for various values of $\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \mu_4, \alpha, \sigma$ and k and the following system measures can be easily study with these probabilities. The following abbreviations are used

MNCO : Mean Number of Customers in the Orbit

MPQL : Mean Number of high priority customers in front of the service station

P_{00} : Probability that the server is idle during the working vacation

P_{10} : Probability that the server is busy with low priority customers during the working vacation

P_{20} : Probability that the server is busy with high priority customers during the working vacation

P_{01} : Probability that the server is idle in non- working vacation

P_{11} : Probability that the server is busy with low priority customers during the non-working vacation

P_{21} : Probability that the server is busy with high priority customers during the non working vacation

a) The probability mass function of Server state in working vacation

Let S(t) be the random variable which represents the server state at time t .

S:	0_{idle}	1_{low}	2_{high}
P:	$\sum_{i=0}^{\infty} p(i,0,0,0)$	$\sum_{i=0}^{\infty}\sum_{j=0}^{k} p(i,j,1,0)$	$\sum_{i=0}^{\infty}\sum_{j=0}^{k} p(i,j,2,0)$

b) The probability mass function of Server state not in working vacation

Let S(t) be the random variable which represents the server state at time t .

S :	0_{idle}	1_{low}	2_{high}
P :	$\sum_{i=0}^{\infty} p(i,0,0,1)$,	$\sum_{i=0}^{\infty}\sum_{j=0}^{k} p(i,j,1,1)$	$\sum_{i=0}^{\infty}\sum_{j=0}^{k} p(i,j,2,1)$

c) The probability mass function of number of customers (low) in the orbit

$$\text{P(no customers in the orbit)} = \sum_{j=0}^{k}\sum_{l=1}^{2}\sum_{m=0}^{1} p(0,j,l,m) + p(0,0,0,0) + p(0,0,0,1)$$

$$\text{P(i customers in the orbit)} = \sum_{j=0}^{k}\sum_{l=1}^{2}\sum_{m=0}^{1} p(i,j,l,m) + p(i,0,0,0) + p(i,0,0,1)$$

d) The Probability mass function of number of high priority customers in the queue.

$$\text{P (no customers in the high priority queue)} = \sum_{i=0}^{\infty}\sum_{l=0}^{2}\sum_{m=0}^{1} p(i,0,l,m)$$

$$\text{P (j customers in the high priority queue)} = \sum_{i=0}^{\infty}\sum_{l=1}^{2}\sum_{m=0}^{1} p(i,j,l,m)$$

e) The Mean number of high priority customers in the queue

$$= \sum_{j=1}^{k} j\left(\sum_{i=0}^{\infty}\sum_{l=1}^{2}\sum_{m=0}^{1} p(i,j,l,m)\right)$$

f) The Mean number of low priority customers in the orbit

$$\text{MNCO} = \left(\sum_{i=0}^{\infty} i\left(\sum_{j=0}^{k}\sum_{l=1}^{2}\sum_{m=0}^{1} p(i,j,l,m) + p(i,0,0,0) + p(i,0,0,1)\right)\right)$$

g) The probability that the orbiting customer (low) is blocked

$$\text{Blocking Probability} = \sum_{i=1}^{\infty}\sum_{j=0}^{k}\sum_{l=1}^{2}\sum_{m=0}^{1} p(i,j,l,m)$$

9. Numerical Study

Table I : Mean number of customers in the orbit and Mean queue length of high Priority queue for $\lambda_1 = 10$ $\lambda_2 = 5$ $\mu_1=20$ $\mu_2=25$ $\mu_3=2$ $\mu_4=5$ $\sigma=100$ k=2 and α varies from 100 to 2000

α	σ	P_{00}	P_{10}	P_{20}	P_{01}	P_{11}	P_{21}	MNCO	MPQL
100	100	0.0338	0.0033	0.0016	0.2622	0.4997	0.1994	1.6129	0.2059
200	100	0.0183	0.0009	0.0004	0.2808	0.4999	0.1996	1.6044	0.2047
300	100	0.0125	0.0004	0.0002	0.2872	0.5000	0.1997	1.6028	0.2045
400	100	0.0095	0.0002	0.0001	0.2905	0.5000	0.1997	1.6022	0.2045
500	100	0.0077	0.0002	0.0001	0.2924	0.5000	0.1997	1.6019	0.2044
600	100	0.0064	0.0001	0.0001	0.2937	0.5000	0.1997	1.6018	0.2044
700	100	0.0055	0.0001	0.0000	0.2947	0.5000	0.1997	1.6017	0.2044
800	100	0.0048	0.0001	0.0000	0.2954	0.5000	0.1997	1.6016	0.2044
900	100	0.0043	0.0000	0.0000	0.2959	0.5000	0.1997	1.6016	0.2044
1000	100	0.0039	0.0000	0.0000	0.2963	0.5000	0.1997	1.6015	0.2044
1100	100	0.0035	0.0000	0.0000	0.2967	0.5000	0.1997	1.6015	0.2044
1200	100	0.0032	0.0000	0.0000	0.2970	0.5000	0.1997	1.6015	0.2044
1300	100	0.0030	0.0000	0.0000	0.2973	0.5000	0.1997	1.6015	0.2044
1400	100	0.0028	0.0000	0.0000	0.2975	0.5000	0.1997	1.6015	0.2044
1500	100	0.0026	0.0000	0.0000	0.2977	0.5000	0.1997	1.6015	0.2044
1600	100	0.0024	0.0000	0.0000	0.2978	0.5000	0.1997	1.6015	0.2044
1700	100	0.0023	0.0000	0.0000	0.2980	0.5000	0.1997	1.6015	0.2044
1800	100	0.0022	0.0000	0.0000	0.2981	0.5000	0.1997	1.6014	0.2044
1900	100	0.0021	0.0000	0.0000	0.2982	0.5000	0.1997	1.6014	0.2044
2000	100	0.0020	0.0000	0.0000	0.2983	0.5000	0.1997	1.6014	0.2044

Table II : Mean number of customers in the orbit and Mean queue length of high Priority queue for $\lambda_1 = 10$ $\lambda_2 = 5$ $\mu_1=20$ $\mu_2=25$ $\mu_3=2$ $\mu_4=5$ $\sigma=1000$ k=2 and α varies from 100 to 2000

α	σ	P_{00}	P_{10}	P_{20}	P_{01}	P_{11}	P_{21}	MNCO	MPQL
100	1000	0.0380	0.0037	0.0018	0.2572	0.4996	0.1996	1.4094	0.2078
200	1000	0.0206	0.0010	0.0005	0.2781	0.4999	0.1999	1.4007	0.2066
300	1000	0.0141	0.0005	0.0002	0.2853	0.5000	0.1999	1.3991	0.2063
400	1000	0.0107	0.0003	0.0001	0.2890	0.5000	0.2000	1.3986	0.2062
500	1000	0.0086	0.0002	0.0001	0.2912	0.5000	0.2000	1.3983	0.2062
600	1000	0.0072	0.0001	0.0001	0.2926	0.5000	0.2000	1.3982	0.2062
700	1000	0.0062	0.0001	0.0000	0.2937	0.5000	0.2000	1.3980	0.2062
800	1000	0.0054	0.0001	0.0000	0.2945	0.5000	0.2000	1.3980	0.2062
900	1000	0.0049	0.0001	0.0000	0.2951	0.5000	0.2000	1.3979	0.2061
1000	1000	0.0044	0.0000	0.0000	0.2956	0.5000	0.2000	1.3979	0.2061
1100	1000	0.0040	0.0000	0.0000	0.2960	0.5000	0.2000	1.3979	0.2061
1200	1000	0.0037	0.0000	0.0000	0.2963	0.5000	0.2000	1.3979	0.2061
1300	1000	0.0034	0.0000	0.0000	0.2966	0.5000	0.2000	1.3979	0.2061
1400	1000	0.0031	0.0000	0.0000	0.2968	0.5000	0.2000	1.3979	0.2061
1500	1000	0.0029	0.0000	0.0000	0.2971	0.5000	0.2000	1.3979	0.2061
1600	1000	0.0027	0.0000	0.0000	0.2972	0.5000	0.2000	1.3979	0.2061
1700	1000	0.0026	0.0000	0.0000	0.2974	0.5000	0.2000	1.3978	0.2061
1800	1000	0.0024	0.0000	0.0000	0.2976	0.5000	0.2000	1.3978	0.2061
1900	1000	0.0023	0.0000	0.0000	0.2977	0.5000	0.2000	1.3978	0.2061
2000	1000	0.0022	0.0000	0.0000	0.2978	0.5000	0.2000	1.3978	0.2061

Table I and Table II show the effect of working vacation rate over the system. As working vacation rate α increase, mean number of customers in the orbit decreases and this model becomes retrial queueing system with non-pre-emptive priority service.

Table III : Mean number of customers in the orbit and Mean queue length of high Priority queue for $\lambda_1 = 10$ $\lambda_2 = 5$ $\mu_1 = 20$ $\mu_2 = 25$ $\mu_3 = 2$ $\mu_4 = 5$ $\alpha = 100$ $k = 6$ and Various values of σ

σ	P_{00}	P_{10}	P_{20}	P_{01}	P_{11}	P_{21}	MNCO	MPQL
10	0.0103	0.0010	0.0005	0.2884	0.4999	0.1999	3.7150	0.2066
20	0.0199	0.0020	0.0010	0.2776	0.4998	0.1998	2.5512	0.2070
30	0.0248	0.0024	0.0012	0.2721	0.4998	0.1997	2.1630	0.2072
40	0.0277	0.0027	0.0013	0.2688	0.4997	0.1997	1.9689	0.2074
50	0.0296	0.0029	0.0014	0.2667	0.4997	0.1997	1.8524	0.2075
60	0.0309	0.0030	0.0015	0.2652	0.4997	0.1997	1.7747	0.2075
70	0.0319	0.0031	0.0015	0.2641	0.4997	0.1997	1.7192	0.2076
80	0.0326	0.0032	0.0016	0.2632	0.4997	0.1997	1.6776	0.2076
90	0.0332	0.0033	0.0016	0.2626	0.4997	0.1997	1.6452	0.2076
100	0.0337	0.0033	0.0016	0.2620	0.4997	0.1997	1.6193	0.2076
200	0.0360	0.0035	0.0017	0.2594	0.4996	0.1996	1.5027	0.2077
300	0.0368	0.0036	0.0018	0.2585	0.4996	0.1996	1.4638	0.2078
400	0.0372	0.0037	0.0018	0.2580	0.4996	0.1996	1.4444	0.2078
500	0.0375	0.0037	0.0018	0.2578	0.4996	0.1996	1.4327	0.2078
600	0.0377	0.0037	0.0018	0.2576	0.4996	0.1996	1.4249	0.2078
700	0.0378	0.0037	0.0018	0.2574	0.4996	0.1996	1.4194	0.2078
800	0.0379	0.0037	0.0018	0.2573	0.4996	0.1996	1.4152	0.2078
900	0.0379	0.0037	0.0018	0.2573	0.4996	0.1996	1.4120	0.2078
1000	0.0380	0.0037	0.0018	0.2572	0.4996	0.1996	1.4094	0.2078
2000	0.0382	0.0038	0.0018	0.2569	0.4996	0.1996	1.3977	0.2078
3000	0.0383	0.0038	0.0018	0.2568	0.4996	0.1996	1.3938	0.2078
4000	0.0384	0.0038	0.0018	0.2568	0.4996	0.1996	1.3919	0.2079
5000	0.0384	0.0038	0.0018	0.2568	0.4996	0.1996	1.3907	0.2079
6000	0.0384	0.0038	0.0018	0.2567	0.4996	0.1996	1.3899	0.2079

Table III shows the impact of retrial rate over the system. Mean number of customers in the orbit decreases as σ increases. When σ is large, values of tables show that this retrial model becomes standard queueing model. Mean number high priority customers (MPQL) increases as k increases

10. Conclusions

It is observed from numerical study that Mean number of low priority customers in the orbit decreases as the retrial rate increases, the probabilities for the server being idle, busy during the working vacation and normal period depend on retrial rate. The various special cases discussed in section 7 are particular cases of this research work. This research work can further be extended by introducing various parameters like negative arrival and second optional services.

References

[1] J.R. Artalejo, A classified bibliography of research on retrial queues Progress in 1990-1999, Top 7, (1999)187-211.

[2] B.D. Choi, Y. Chang. Single server retrial queues with priority calls, Math. and Comp. Modelling, 30 (3- 4) (1999) 7 – 32.

[3] G.I. Falin, J.R. Artalejo, M. Martin, On the single server retrial queue with priority customers, Queueing systems 14 (1993) 439-455.

[4] A. Gomez-Corral, A Bibliographical guide to the analysis of retrial queues through matrix analytic technique, Annals of Operationas, 141(2006)163-191.

[5] G. Ayyappan, A. Muthu Ganapathi Subramanian, Gopal sekar. M/M/1 Retrial Queueing System with Non-pre-emptive priority Service and Single Vacation – Exhaustive Service, Pacific Asian J. of Math., 3 (1-2) (2009) 307-322.

[6] W. Liu, X. Xu, N. Tian, Some results on the M/M/1 queue with working vacation, Operation Research Letters, 35 (5)(2007) 595-600.

[7] L.D. Servi, S.G. Finn, M/M/1 queue with working vacation, Performance evaluations, 50 (1) (2002) 41-52.

[8] N. Tian, X. Zhao, Wang.K, The M/M/1 queue with single working vacation, Int. J. of Information and Management Science, 19 (4)(2008) 621-634.

[9] Tien Van Do, M/M/1 retrial queue with working vacation, Acta Informatica, 47 (1) (2009) 67-75.

[10] D. Wu, H. Takagi, M/G/1 queue with Multiple working vacation, Performance Evaluation, 63 (7) (2006) 654-681.