f-Morphisms in Fuzzy Graphs

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Abstract

We define a new f-morphism between two fuzzy graphs. This f-morphism is defined on two scale factors, one based on the scaling of weights of the vertices and the other on the scaling of weights of the edges. We prove that, the relation "f-morphic" is an equivalence relation in the collection of fuzzy graphs. We prove that under this f-morphism, the image of an effective edge is an effective edge, the images of fuzzy tree, fuzzy cycle remain as fuzzy tree and fuzzy cycle. We also prove if G_1, G_2 are two fuzzy graphs which are f-morphic then $\overline{G}_1$ and $\overline{G}_2$ are also f-morphic.

Key words : f-morphism, fuzzy graphs.

Mathematics Subject Classification : **05C72, 03E72.**

1. Introduction

The following definitions of isomorphism and weak isomorphism of Fuzzy Graphs are due to Mordeson [1] :

Definition 1.1

Let G = (V,X) and $G'=(V', X')$ be graphs. Let μ be a fuzzy subset of V and ρ be a fuzzy subset of V xV such that (μ, ρ) is a partial fuzzy sub graph of G. Let (μ', ρ') be partial fuzzy sub graph of G'. Let f be a One-to-one Function of V on to V'. Then,

(i) f is called a [weak] Vertex –isomorphism of (μ,ρ) onto (μ',ρ') if and only if [$\forall v \in V$, $\mu(v) \leq \mu'(f(v))$ and supp (μ') = (f(supp(μ))] $\mu(v) = \mu'(f(v))$.

(ii) f is called a [weak] line-isomorphism of (μ,ρ) onto ((μ', ρ') if and only if $\forall (u, v)\in X$, [$\rho(u, v) \leq \rho'(f(u), f(v))$ and supp $(\rho') = \{(f(u), f(v)) / (u, v) \in supp(\rho)\}$] $\rho(u, v) = \rho'(f(u), f(v))$.

In case (i), the graphs are exactly the same and in case (ii), the class of graphs which are weakly isomorphic to a given fuzzy graph is a heterogeneous collection of fuzzy graphs. We try to bridge the gap between these two extreme definitions and try to define a morphism that lies between these two definitions. Our definition of f-morphism partitions the family of fuzzy graphs into a set of equivalence classes. This f-morphism is defined on two scale factors, one for the vertices and the other for the edges. When this scale factors become unity we get our usual isomorphism of fuzzy graphs [3]. In section 2, we list preliminary definitions. In section 3 we define the f-morphism and give examples. In section 4, we derive results using the f-morphism. We conclude with section 5.

2. Preliminaries

We list only important definitions. For more details refer [2, 4]

Definition 2.1

A fuzzy Graph $G = (V,\mu ,\rho)$ is a nonempty set V together with a pair of functions $\mu : V \rightarrow [0, 1]$ and $\rho : V \times V \rightarrow [0, 1]$ such that for all $x,y \in V$, we have $\rho(x,y) \leq \mu(x) \wedge \mu(y)$. For simplicity, we denote the fuzzy graph by $G = (\mu ,\rho)$.Here $\wedge$ denotes the minimum.

Definition 2.2

We define the sets supp(μ) and supp(ρ) as follows

$\text{supp}(\mu) = \{ x \in V \mid \mu(x) > 0 \}$

$\text{supp}(\rho) = \{(x,y) \in V \times V \mid \rho(x,y) > 0\}$

Definition 2.3

The fuzzy graph $H = (v,\tau)$ is called a partial fuzzy subgraph of $G = (V,\mu ,\rho)$ if $v \subseteq \mu$ and $\tau \subseteq \rho$.

Definition 2.4

A path in a fuzzy graph (μ ,ρ) is a sequence of distinct vertices $x_0, x_1, x_2, \ldots\ldots, x_n$ such that $\rho(x_{i-1},x_i) > 0$, $1 \leq i \leq n$. The strength of the path is defined as $\wedge\{\rho(x_{i-1},x_i) \mid i = 1,2\ldots..n\}$.

Definition 2.5

$G = (\mu ,\rho)$ is called a tree if (supp(μ),supp(ρ)) is a tree. $G = (\mu ,\rho)$ is called a fuzzy tree if (μ ,ρ) has a fuzzy spanning subgraph (μ,v) which is a tree such that $\forall$ $(u,v) \in$ supp(μ) but not in supp(v) we have

$\rho(u,v) < v^{\infty}(u,v)$.

i.e. there exists a path in (μ,v) between u and v whose strength is greater than $\rho(u,v)$.

Definition 2.6

$G = (\mu, \rho)$ is called a cycle if (supp (μ), supp (ρ)) is a cycle. $G = (\mu, \rho)$ is called a fuzzy cycle iff (supp(μ),supp(ρ)) is a cycle and there does not exist a unique $(x,y) \in \mathrm{supp}(\rho)$ such that $\rho(x,y) = \Lambda\{\rho(u,v) \mid (u,v) \in \mathrm{supp}(\rho)\}$.

Definition 2.7

The complement of a fuzzy graph $G = (V,\mu, \rho)$ is defined as $\bar{G} =(V, \mu, \bar{\rho})$ where $\bar{\rho}(x,y) = \mu(x) \Lambda \mu(y) - \rho(x,y)$.

Definition 2.8

An M-strong fuzzy graph is a Fuzzy graph in which, $\rho(x,y) = \mu(x) \wedge \mu(y)$ for every nonzero edge (x,y)

3. f-Morphism

Definition 3.1

Let (G_1, μ_1, ρ_1) and (G_2, μ_2, ρ_2) be two fuzzy Graphs, with vertex sets V_1 and V_2 . A bijective function $f : V_1 \rightarrow V_2$ is called a fuzzy morphism or f-morphism if there exist real numbers k_1 and $k_2 > 0$ such that

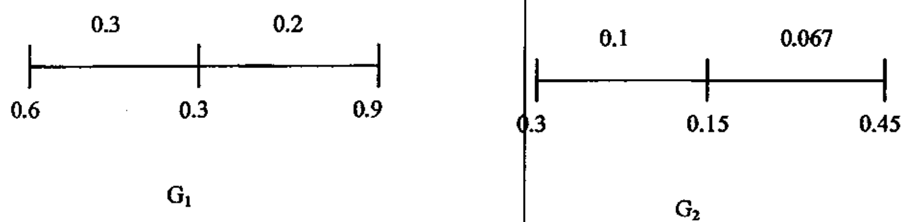
(i) $\mu_2 (f(u))=k_1 \mu_1 (u) \ \forall u \in V_1$

(ii) $\rho_2 (f(u),f(v))=k_2 \rho_1 (u, v) \ \forall u, v\in V_1$.

In such a case f will be called as (k_1 , k_2)-f-morphism from G_1 to G_2.

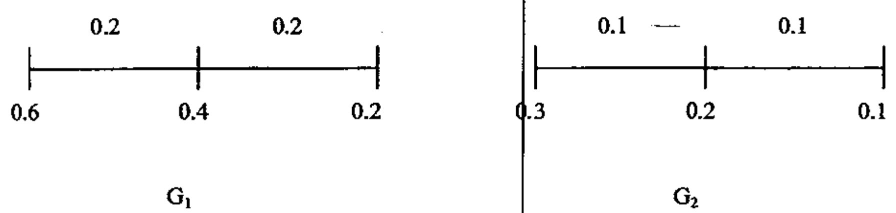
Note: If $k_1 = k_2 = k$ then we call f as k-morphism. If $k_1 = k_2 = k=1$ then we get our usual fuzzy isomorphism.

Example 3.2



Here we take $k_1 = 1/2$ and $k_2 = 1/3$

Example 3.3



Here we take $k_1 = k_2 = 1/2$.

Given the value $k_1 > 0$, We find the threshold values for k_2, so that the condition for the fuzzy graph, $\rho_2(f(u),f(v)) \leq \mu_2(f(u)) \wedge \mu_2(f(v)) \quad \forall u,v \in V_1$ is satisfied. We have two cases.

Case (i) Assume $k_2 \leq k_1$

Then, $\rho_2(f(u),f(v)) = k_2\,\rho_1(u,v) \quad \forall u,v \in V_1$

$$\leq k_2\ (\mu_1(u) \wedge \mu_1(v))$$

$$\leq k_1\ (\mu_1(u) \wedge \mu_1(v)) \text{ by assumption}$$

$$\leq k_1\ (\mu_1(u)) \wedge k_1(\mu_1(v))$$

$$\leq \mu_2(f(u)) \wedge \mu_2(f(v))$$

There fore, if $0 \leq k_2 \leq k_1$, then for each k_2 we have a (k_1, k_2)-f-morphism from G_1 to G_2.

Case (ii) Assume $k_2 > k_1$

Now, $\rho_2(f(u),f(v)) = k_2\,\rho_1(u,v) \quad \forall u,v \in V_1$.

Also $\rho_2(f(u),f(v)) \leq \mu_2\,(f(u) \wedge \mu_2(f(v)) \quad \forall u,v \in V_1$

$$\Rightarrow \quad k_2\,\rho_1(u,v) \leq k_1\,(\mu_1(u)) \wedge k_1(\mu_1(v))$$

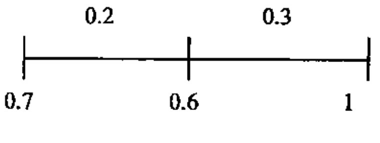
$$\Rightarrow \quad k_2 \leq \frac{k_1\,\mu_1(u) \wedge k_1\mu_1(v)}{\rho_1(u,v)}$$

$$\Rightarrow \quad k_2 \leq \frac{k_1\,\mu_1(u)}{\rho_1(u,v)} \wedge \frac{k_1\,\mu_1(v)}{\rho_1(u,v)}$$

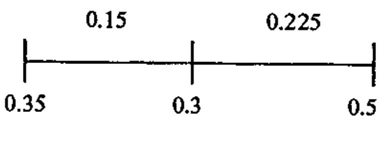
This inequality is true for each edge (u,v). Therefore, k_2 must be chosen such that

$0 < k_2 \leq \text{Min} \left\{ \frac{k_1\mu_1(u)}{\rho_1(u,v)} \right\}$ for u $\in V_1$ and (u,v) is an edge in G_1

Example 3.4



G_1



G_2

Here $k_1 = \frac{1}{2}$ and $k_2 = \frac{3}{4}$

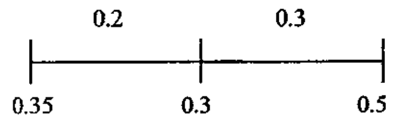
Assuming $k_1 = \frac{1}{2}$ the value of k_2 must satisfy the condition

$$k_2 \leq \text{Min} \left\{ \frac{(.5)(.7)}{.2}, \frac{(.5)(.6)}{.2}, \frac{(.5)(.6)}{.3}, \frac{(.5)(1)}{.3} \right\}$$

$\leq$ Min {1.75,1.5,1,1.66}

≤ 1

Taking $k_2 = 1$ we have the second graph



G_2

4. Theorems on f-Morphism

Theorem 4.1

The relation "f-morphic" is an equivalence relation in the collection of fuzzy graphs.

Proof

Consider the collection of all fuzzy graphs. For two fuzzy graphs G_1 and G_2 define the relation $G_1 \approx G_2$ if there exists a (k_1 , k_2)-f-morphism from G_1 to G_2. We prove $\approx$ is an equivalence relation.

Clearly $G_1 \approx G_1$ by taking both k_1 and k_2 to be one.

If $G_1 \approx G_2$ then there exists a (k_1 , k_2)-morphism from G_1 to G_2 for some k_1 and k_2 .This induces a $\left(\frac{1}{k_1}, \frac{1}{k_2}\right)$ morphism from G_2 to G_1 and hence $G_2 \approx G_1$.

Let $G_1 \approx G_2$ and $G_2 \approx G_3$. then, there exists a (k_1 , k_2)-morphism from G_1 to G_2 (say f) for some k_1 and k_2 and there exists a (k_3 , k_4)-morphism from G_2 to G_3 (say g)for some k_3 and k_4

We have $\mu_2(f(x)) = k_1\, \mu_1(x) \quad \forall x \in V_1$,

$\rho_2(f(x), f(y)) = k_2\, \rho_1(x,y) \quad \forall\, x, y \in V_1$ and

$\mu_3(g(u)) = k_3\, \mu_2(u) \quad \forall u \in V_2$

$\rho_3(g(u), g(v)) = k_4\, \rho_2(u,v) \quad \forall\, u, v \in V_2$.

Therefore, $\mu_3\, (g\,(u)) = k_3\, \mu_2\, (u)$

$$\Rightarrow \mu_3(g(u)) = k_3\, \mu_2(f(x))$$
$$= \quad k_3\, k_1\, \mu_1(x) \quad \forall x \in V_1 \text{ and,}$$
$$\rho_3(g(u), g(v)) = k_4\, \rho_2(u,v) \quad \forall\, u, v \in V_2$$
$$\Rightarrow \quad \rho_3(g(u), g(v)) = k_4\, \rho_2(f(x), f(y)) \quad \textit{where } u = f(x) \textit{ and } v = f(y)$$
$$= k_4\, k_2\, \rho_1(x,y) \quad \forall\, x, y \in V_1$$

There exists a $(k_3\, k_1,\ k_2\, k_4)$ f-momorphism from G_1 to G_3

Hence $G_1 \approx G_3$.

Definition 4.2

An edge (x,y) in a fuzzy graph is called an effective edge if $\rho(x,y) = \mu(x) \wedge \mu(y)$.

Theorem 4.3

Let G_1 be k- f-morphic to G_2. Then, the image of an effective edge is an effective edge.

Proof

Let (x,y) be a strong edge in G_1 . Then, $\rho_1(x,y)=\mu_1(x)\wedge\mu_1(y)$.

Now, $\rho_2(f(x),f(y))=k\ \rho_1(x,y)$

$=k\ (\mu_1(x)\wedge\mu_1(y))$

$=k\ \mu_1(x)\wedge k\mu_1(y)$

$= \mu_2(f(x))\wedge\ \mu_2(f(y))$

$\Rightarrow$ (f(x),f(y)) is an effective edge

Corollary 4.4

If G_1 is k-f-morphic to G_2 for some k and if G_1 is M-strong then G_2 is also M-strong.

Theorem 4.5

Let G_1 , G_2 be two fuzzy graphs which are (k_1 , k_2)-f-morphic. If the image of an effective edge in G_1 is in an effective edge in G_2 then $k_1 = k_2$.

Proof

Let (x,y) be an effective edge in G_1 such that (f(x),f(y)) is also an effective edge in G_2. Then, $\rho_1(x,y)= \mu_1(x)\wedge\mu_1(y)$ and $\rho_2(f(x),f(y))=\mu_2(f(x))\wedge\mu_2(f(y))$

Also, $\rho_2(f(x),f(y))=k_2\ \rho_1(x,y)$

$$\begin{aligned}\Rightarrow\ k_2\ \rho_1(x,y) &= \mu_2(f(x))\wedge\mu_2(f(y))\\ &= k_1\ \mu_1(x)\ \wedge k_1\mu_1(y)\\ &= k_1(\mu_1(x)\ \wedge\mu_1(y))\\ &= k_1\ \rho_1(x,y)\end{aligned}$$

$\Rightarrow \quad k_1=k_2$.

Theorem 4.6

Let G_1, G_2 be two fuzzy graphs such that G_1 is f-morphic to G_2. Then,

(i) G_1 is a fuzzy tree $\Rightarrow$ G_2 is a fuzzy tree.

(ii) G_1 is a fuzzy cycle $\Rightarrow$ G_2 is a fuzzy cycle.

(iii) G_1 is a fuzzy regular $\Rightarrow$ G_2 is a fuzzy regular

Proof

(i) Under f-morphiosm adjacency of vertices is maintained and therefore a tree will remain a tree. Since the weights of edges are uniformly scaled up or scaled down,the strengths of paths are also scaled up or scaled down by the same factor. Let (G_1, μ_1, ρ_1) be a fuzzy tree. Then, (G_1, μ_1, ρ_1) has a fuzzy spanning subgraph (μ,v_1) which is atree such thar $\forall(u, v) \in$ supp (p_1) \ supp (v), we have $\rho_1^{\infty}(u,v) < v_1^{\infty}(u,v)$. Now,

$$\rho_2^{\infty}(f(u), f(v)) = k_2\, \rho_1^{\infty}(u,v)$$

$$< k_2 v_1^{\infty}(u,v)$$

$$= v_2^{\infty}(f(u), f(v))$$

Therefore, the image of a fuzzy tree is again a fuzzy tree.

(ii) Let (G_l, μ_1, ρ_1) be a fuzzy cycle. Then (supp (μ_1) supp (ρ_1) is acycle and there does not exist a unique (x,y) $\in$ supp (ρ_1) such that $\rho_1(x,y) = \wedge\{\rho_1(u,v)/(u,v) \in \sup p(\rho_1)\}$ Correspondingly of the image of(supp (μ_1), supp (ρ_1) is also a cycle as adjacency is maintained. Also there does not exist a unique edge (f(x),f(y)) such that

$$\{\rho_2(f(x), f(y)\} = \wedge\{k_2\rho_1(u,v)/(u,v) \in \sup p(\rho_1)\}$$

$$= k_2(\wedge\{\rho_1(u,v)/(u,v) \in \sup p(\rho_1)\})$$

$$= k_2\, \rho_1(x,y)$$

(3) Let G_1 be a fuzzy regular.

$\Rightarrow \sum_v \rho_1(u,v)$ **is same for all vertices u**

$\Rightarrow \sum_v k_2\rho_1(u,v)$ **is same for all vertices u**

$\Rightarrow \sum_v \rho_2(f(u), f(v))$ **is same for vertices u**

$\Rightarrow$ **G_2 is fuzzy regular.**

Theorem 4.7

If G_1, G_2 are two fuzzy graphs which are k-f-morphic then $\overline{G_1}$ and $\overline{G_2}$ are also k-f morphic.

Proof

The complement $(\overline{G_1}, \overline{\mu_1}, \overline{\rho_1})$ is defined as where $\overline{\mu_1} = \mu_1$ and

$\overline{\rho_1}$ (x,y) = $\mu_1(x) \wedge \mu_1(y) - \rho_1(x,y)$.

Now, $\overline{\rho_2}$ (f(u), f(v))= $\mu_2(f(u)) \wedge \mu_2(f(v)) - \rho_2(f(u), f(v))$

$$= k\mu_1(u) \wedge k\mu_1(v) - k\rho_1(u,v)$$

$$= k(\mu_1(u) \wedge \mu_1(v) - \rho_1(u,v))$$

$$= k\overline{\rho}_1(u,v).$$

Hence $\overline{G}_1$ and $\overline{G}_2$ are also k-f-morphic.

5. Conclusion

We have introduced the concept of f-Morphism in fuzzy graphs and have derived results. We, explore these concepts with respect to the matrices associated with the fuzzy graphs in our next paper. We also study the relation between f- morphism of two fuzzy graphs and the f-Morphism of their fuzzy line graphs.

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