A Note on a New Three Variable Analogue of Hermite Polynomials of II Kind

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Abstract

The present paper is a study of a new three variable analogue of Hermite polynomials $H_n(x,y,z)$ whose two variables version seems more natural than that of Hemite polynomials of two variables defined and studied by M.A. Khan and G.S. Abukhammash [1].

Keywords: Generating functions, recurrence relations, Rodrigues formula, relationship with Hermite polynomials of one variable, some special properties and expansion of Legendre polynomials in terms of Hermite polynomials of three variables.

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1. Hermite Polynomials of Three Variables

Hermite polynomials of three variable $H_n(x,y,z)$ is defined as follows:

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-n)_{2r}(-1)^r H_{n-2r}(x,y)\ z^{n-2r}\left[\left(x^2+y^2\right)\left(1-z^2\right)+z^2\right]^r}{r!} \tag{1.1}$$

where H_n(x,y) is Hermite polynomial of two variables [2]. The definition (1.1) can also be written as

$$H_n(x,y,z) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{s=0}^{\left[\frac{n}{2}-r\right]} \frac{(-1)^{r+s} n! H_{n-2r-2s}(xy) x^{2r} y^{2s} z^{n-2r-2s}}{r!s!(n-2r-2s)!} \tag{1.2}$$

where $H_n(x)$ is the well-known Hermite polynomial of one variable.

The definition (1.2) is equivalent to the following explicit representation of $H_n(x,y,z)$:

$$H_n(x,y,z)=(2xyz)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{r=0}^{\left[\frac{n}{2}-k\right]} \sum_{s=0}^{\left[\frac{n}{2}-k-r\right]} \frac{(-n)_{2k+2r+2s}\left(-\frac{1}{x^2y^2}\right)^s\left(-\frac{1}{y^2z^2}\right)^k\left(-\frac{1}{x^2z^2}\right)^r}{2^{2k+2r+2s}k!r!s!} \quad (1.3)$$

In terms of triple hypergeometric function, Hermite polynomials of three variables can be written as

$$H_n(x,y,z)=(2xyz)^n F^{(3)}\left[\begin{matrix} -\frac{n}{2},-\frac{n}{2}+\frac{1}{2}::-;-;-:-;-;-; \\ -::-;-;-:-;-;-; \end{matrix} -\frac{1}{x^2y^2},-\frac{1}{y^2z^2},-\frac{1}{z^2x^2}\right] \quad (1.4)$$

Where for right hand side of (1.4), it may be recalled that the definition of a general triple hypergeometric $F^{(3)}[x, y, z]$ (ef. H.M. Srivastava and H.L. Manocha [4], p. 428) is defined as:

$$F^{(3)}[x, y, z] \equiv F^{(3)}\left[\begin{matrix}(a)::(b);\ (b')\ ;\ (b'')\ :\ (d)\ ;\ (d')\ ;\ (d'')\ ; \\ (a)::(g);\ (g')\ ;\ (g'')\ :\ (h)\ ;\ (h')\ ;\ (h'')\ ;\end{matrix} x,y,z\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \Lambda(m,n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^P}{p!}, \quad (1.5)$$

where, for convenience

$$\Lambda(m,n,p)=\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{p+n}\prod_{j=1}^{B''}(b''_j)_{p+m}\prod_{j=1}^{D}(d_j)_m\prod_{j;1}^{D'}(d'_j)_n\prod_{j=1}^{D''}(d''_j)_p}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p} \quad (1.6)$$

where (a) abbreviates, the array of a parameters $a_1,a_2,\ldots a_A$, with similar interpretations for (b), (b′), (b″), etc. The triple hypergeometric series in (1.5) converges absolutely when

$$\left.\begin{matrix} 1+E+G+G''+H-A-B-B''-C\geq 0 \\ 1+E+G+G'+H'-A-B-B'-C'\geq 0 \\ 1+E+G'+G''+H''-A-B'-B''-C''\geq 0 \end{matrix}\right\} \quad (1.7)$$

where the equalities hold true for suitable constrained values of $|x|$, $|y|$ and $|z|$.

The paper contains Generating functions, recurrence relations, Rodrigues formula and relationship with Hermite polynomials of one variable, some special properties and expansion of Legendre polynomials in terms of Hermite polynomials of three variables.

2. Generating Function for $H_n(x,y,z)$

Some generating functions for Hermite polynomials of three variables $H_n(x,y,z)$ are as follows:

$$e^{2xyzt-(x^2+y^2+z^2)t^2}=\sum_{n=0}^{\infty}\frac{H_n(x,y,z)t^n}{n!} \tag{2.1}$$

$$\sum_{n=0}^{\infty}\frac{(c)_n H_n(x,y,z)t^n}{n!}\cong(1-2xyzt)^{-c}$$

$$\times F^{(3)}\left[\begin{matrix}\frac{c}{2},\frac{c}{2}+\frac{1}{2}::-;-;-:-;-;-:-;-;- \\ -::-;-;-:-;-;-:-;-;-\end{matrix} \; -\frac{4x^2t^2}{(1-2xyzt)^2},-\frac{4y^2t^2}{(1-2xyzt)^2},-\frac{4z^2t^2}{(1-2xyzt)^2}\right] \tag{2.2}$$

$$\sum_{n=0}^{\infty}\frac{H_{n+k}(x,y,z)t^n}{n!}$$

$$=e^{2xyzt-(x^2+y^2+z^2)t^2}\sum_{n=0}^{\left[\frac{k}{2}\right]}\frac{(-k)_{2n}H_{k-2n}\left[xyz-t(x^2+y^2+z^2)\right]\left[1-(x^2+y^2+z^2)\right]^n}{n!} \tag{2.3}$$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{H_{n+r+s}(x,y,z)t^n u^r}{r!\,n!}=e^{2xyzt-(x^2+y^2+z^2)t^2}.e^{2\left[xyz-t(x^2+y^2+z^2)\right]u-(x^2+y^2+z^2)u^2}$$

$$\times\sum_{n=0}^{\left[\frac{s}{2}\right]}\frac{(-s)_{2n}H_{s-2n}\left[xyz-t(x^2+y^2+z^2)-u(x^2+y^2+z^2)\right]\left[1-(x^2+y^2+z^2)\right]^n}{n!} \tag{2.4}$$

3. Special properties

In this section we obtain some special properties for Hermite polynomials of three variable $H_n(x,y,z)$ as given below:

Consider the identity

$$e^{2xyzt-(x^2+y^2+z^2)t^2}=e^{2yz(xt)-(xt)^2}.e^{-y^2t^2}.e^{-z^2t^2}$$

or

$$\sum_{n=0}^{\infty}\frac{H_n(x,y,z)t^n}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-1)^{r+s}H_{n-2r-2s}(yz)x^{n-2r-2s}y^{2r}z^{2s}t^n}{r!\,s!\,(n-2r-2s)!}$$

Equating the coefficients of t^n, we get

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-1)^{r+s}n!\,H_{n-2r-2s}(yz)x^{n-2r-2s}y^{2r}z^{2s}}{r!\,s!\,(n-2r-2s)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-n)_{2r+2s}(-1)^{r+s}}{r!s!}H_{n-2r-2s}(yz)x^{n-2r-2s}y^{2r}z^{2s} \qquad (3.1)$$

Similarly, by considering different identities we obtain the following special properties for $H_n(x,y,z)$:

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-1)^{r+s}n!H_{n-2r-2s}(zx)x^{2r}y^{n-2r-2s}z^{2s}}{r!s!(n-2r-2s)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-n)_{2r+2s}(-1)^{r+s}}{r!s!}H_{n-2r-2s}(zx)x^{2r}y^{n-2r-2s}z^{2s} \qquad (3.2)$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-1)^{r+s}n!H_{n-2r-2s}(xy)x^{2r}y^{2s}z^{n-2r-2s}}{r!s!(n-2r-2s)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{n}{2}-r\right]}\frac{(-n)_{2r+2s}(-1)^{r+s}}{r!s!}H_{n-2r-2s}(xy)x^{2r}y^{2s}z^{n-2r-2s} \qquad (3.3)$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n!H_{n-2r}(x)(yz)^{n-2r}\left[x^2+y^2+z^2(1-y^2)\right]^r}{r!(n-2r)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-n)_{2r}(-1)^r H_{n-2r}(x)(yz)^{n-2r}\left[x^2+y^2+z^2(1-y^2)\right]^r}{r!} \qquad (3.4)$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n!H_{n-2r}(y)(xz)^{n-2r}\left[x^2(1-z^2)+y^2+z^2\right]^r}{r!(n-2r)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-n)_{2r}(-1)^r H_{n-2r}(y)(zx)^{n-2r}\left[x^2(1-z^2)+y^2+z^2\right]^r}{r!} \qquad (3.5)$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n!H_{n-2r}(z)(xy)^{n-2r}\left[x^2+y^2(1-x^2)+z^2\right]^r}{r!(n-2r)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-n)_{2r}(-1)^r H_{n-2r}(z)(xy)^{n-2r}\left[x^2+y^2(1-x^2)+z^2\right]^r}{r!} \tag{3.6}$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n! H_{n-2r}(x,y)z^{n-2r}\left[(x^2+y^2)(1-z^2)+z^2\right]^r}{r!(n-2r)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-n)_{2r}(-1)^r H_{n-2r}(x,y)z^{n-2r}\left[(x^2+x^2)(1-z^2)+z^2\right]^r}{r!} \tag{3.7}$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n! H_{n-2r}(y,z)x^{n-2r}\left[x^2+(1-x^2)(y^2+z^2)\right]^r}{r!(n-2r)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-n)_{2r}(-1)^r H_{n-2r}(y,z)x^{n-2r}\left[x^2+(1-x^2)(y^2+z^2)\right]^r}{r!} \tag{3.8}$$

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n! H_{n-2r}(z,x)y^{n-2r}\left[y^2+(1-y^2)(x^2+z^2)\right]^r}{r!(n-2r)!}$$

or,

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-n)_{2r}(-1)^r H_{n-2r}(z,x)y^{n-2r}\left[y^2+(1-y^2)(x^2+z^2)\right]^r}{r!} \tag{3.9}$$

$$H_n(x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^s n! H_{n-r-2s}\left(\frac{xy}{2}\right)H_r\left(\frac{yz}{2}\right)x^r y^{2s} z^{n-r-2s}}{r!s!(n-r-2s)!} \tag{3.10}$$

$$H_n(x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^s n! H_{n-r-2s}\left(\frac{xy}{2}\right)H_r\left(\frac{zx}{2}\right)x^{n-r-2s} y^r z^{2s}}{r!s!(n-r-2s)!} \tag{3.11}$$

$$H_n(x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^s n! H_{n-r-2s}\left(\frac{xy}{2}\right)H_r\left(\frac{yz}{2}\right)x^{2s} y^{n-r-2s} z^r}{r!s!(n-r-2s)!} \tag{3.12}$$

$$H_n(x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}(1,y,z)(2yz)^r(1-x)^{r+s}(1+x)^s}{r!s!(n-r-2s)!} \tag{3.13}$$

$$H_n(x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}(x,1,z)(2xz)^r(1-y)^{r+s}(1+y)^s}{r!s!(n-r-2s)!} \tag{3.14}$$

$$H_n(x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}(x,y,1)(2xy)^r(1-z)^{r+s}(1+z)^s}{r!s!(n-r-2s)!} \tag{3.15}$$

$$H_n(x_1+x_2,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{n!H_{n-r-2s}(x_1,y,z)H_r(x_2,y,z)[(y^2+z^2)-2x_1x_2]^s}{r!s!(n-r-2s)!} \tag{3.16}$$

$$H_n(x,y_1+y_2z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{n!H_{n-r-2s}(x,y_1,z)H_r(x,y_2,z)[(x^2+z^2)-2y_1y_2]^s}{r!s!(n-r-2s)!} \tag{3.17}$$

$$H_n(x,y,z_1+z_2)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{n!H_{n-r-2s}(x,y,z_1)H_r(x,y,z_2)[(x^2+y^2)-2z_1z_2]^s}{r!s!(n-r-2s)!} \tag{3.18}$$

$$H_n(x_1+x_2,y_1+y_2,z_1+z_2)=\sum_{r=0}^{n}\sum_{s=0}^{(n-r)}\sum_{p=0}^{\left[\frac{n-r-s}{2}\right]}\frac{(-1)^p 2^{s+p} n!H_{n-r-s-2p}(x_1,y_1,z_1)}{r!s!p!}$$

$$\times\frac{H_r(x_2,y_2,z_2)[x_1(y_1z_2+y_2z_1+y_2z_2)+x_2(y_1z_2+y_2z_1+y_1z_1)]^s(x_1x_2+y_1y_2+z_1z_2)^p}{(n-r-s-2p)!} \tag{3.19}$$

$$H_n(\lambda x,y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}(x,y,z)(1-\lambda)^{r+s}(1+\lambda)^s(2yz)^r x^{r+2s}}{r!s!(n-r-2s)!} \tag{3.20}$$

$$H_n(x,\mu y,z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}(x,y,z)(1-\mu)^{r+s}(1+\mu)^s(2zx)^r y^{r+2s}}{r!s!(n-r-2s)!} \tag{3.21}$$

$$H_n(x,y,\nu z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}(x,y,z)(1-\nu)^{r+s}(1+\nu)^s(2xy)^r z^{r+2s}}{r!s!(n-r-2s)!} \tag{3.22}$$

$$H_n(\lambda x,\mu y,\nu z)=\sum_{r=0}^{n}\sum_{s=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^r n!H_{n-r-2s}[(1-\lambda)x,(1-\mu)y,(1-\nu)z]}{r!s!}$$

$$\times \frac{[(1-\lambda)(1-\mu)(1-\nu)-\lambda\mu\nu]^r \left[(1-2\lambda)x^2+(1-2\mu)y^2+(1-2\nu)z^2\right]^s (2xyz)^r}{(n-r-2s)!} \tag{3.23}$$

4. Recurrence Relations

The following recurrence relations hold for $H_n(x,y,z)$:

$$\frac{\partial}{\partial x} H_n(x,y,z) = 2nyz\ H_{n-1}(x,y,z) - 2n(n-1)x\,H_{n-2}(x,y,z) \tag{4.1}$$

$$\frac{\partial}{\partial y} H_n(x,y,z) = 2nxz\ H_{n-1}(x,y,z) - 2n(n-1)y\,H_{n-2}(x,y,z) \tag{4.2}$$

$$\frac{\partial}{\partial z} H_n(x,y,z) = 2nxy\ H_{n-1}(x,y,z) - 2n(n-1)z\,H_{n-2}(x,y,z) \tag{4.3}$$

$$H_n(x,y,z) = 2xyz\ H_n(x,y,z) - 2(n-1)(x^2+y^2+z^2)H_{n-2}(x,y,z) \tag{4.4}$$

5. Relation Between $H_n(x,y,z)$ and $H_n(x)$

We obtain the following relation between $H_n(x,y,z)$ and $H_n(x)$:

$$H_n(x,y,z) = (x^2+y^2+z^2)^{\frac{n}{2}} H_n\left(\frac{xyz}{\sqrt{x^2+y^2+z^2}}\right) \tag{5.1}$$

Now

$$H_n(-x,y,z) = (-1)^n H_n(x,y,z) \tag{5.2}$$

For x=0, (3.4) reduces to

$$H_n(0,y,z) = (yz)^n H_n(0) \tag{5.3}$$

And for y=0, (3.5) reduces of

$$H_n(x,0,z) = (zx)^n H_n(0) \tag{5.4}$$

Also for z=0, (3.6) reduces to

$$H_n(x,y,0) = (xy)^n H_n(0) \tag{5.5}$$

But

$$H_{2n}(0) = (-1)^n 2^{2n}\left(\frac{1}{2}\right)_n ;\ H_{2n+1}(0) = 0. \tag{5.6}$$

So using (5.6) in (5.3), (5.4) and (5.5), we obtain

$$\left.\begin{array}{l} H_{2n}(0,y,z)=(-1)^n(2yz)^{2n}\left(\frac{1}{2}\right)_n \\ H_{2n+1}(0,y,z)=0 \end{array}\right\} \quad (5.7)$$

and

$$\left.\begin{array}{l} H_{2n}(x,0,z)=(-1)^n(2xz)^{2n}\left(\frac{1}{2}\right)_n \\ H_{2n+1}(x,0,z)=0 \end{array}\right\} \quad (5.8)$$

also

$$\left.\begin{array}{l} H_{2n}(x,y,0)=(-1)^n(2xy)^{2n}\left(\frac{1}{2}\right)_n \\ H_{2n+1}(x,y,0)=0 \end{array}\right\} \quad (5.9)$$

For $x=y=z=0$, (3.4) reduces to

$$H_n(0,0,0)=0=H_n(0) \quad (5.10)$$

which in the light of (5.6), gives

$$H_{2n}(0,0,0)=(-1)^n 2^{2n}\left(\frac{1}{2}\right)_n=H_{2n}(0) \quad (5.11)$$

$$H_{2n+1}(0,0,0)=0=H_{2n+1}(0) \quad (5.12)$$

Now

$$H_n(x,y,z)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r\, n!\, H_{n-2r}(x)(yz)^n[x^2+y^2+z^2(1-y^2)]^r}{r!(n-2r)} \quad (5.13)$$

and

$$H_n'(x)=\frac{d}{dx}H_n(x).$$

If we denote

$$\left[\frac{\partial}{\partial x}H_n(x,y,z)\right]_{x=0} \; by \; \frac{\partial}{\partial x}H_n(0,y,z)$$

then

$$\left.\begin{aligned} &\frac{\partial}{\partial x}H_{2n}(0,y,z)=0=H'_{2n}(0)\\ &\frac{\partial}{\partial x}H_{2n+1}(0,y,z)=(-1)^n(2yz)^{2n+1}\left(\frac{3}{2}\right)_n=H'_{2n+1}(0)\end{aligned}\right\} \tag{5.14}$$

and

$$\left.\begin{aligned} &\frac{\partial}{\partial y}H_{2n}(x,0,z)=0=H'_{2n}(0)\\ &\frac{\partial}{\partial y}H_{2n+1}(x,0,z)=(-1)^n(2xz)^{2n+1}\left(\frac{3}{2}\right)_n=H'_{2n+1}(0)\end{aligned}\right\} \tag{5.15}$$

also

$$\left.\begin{aligned} &\frac{\partial}{\partial z}H_{2n}(x,y,0)=0=H'_{2n}(0)\\ &\frac{\partial}{\partial z}H_{2n+1}(x,y,0)=(-1)^n(2xy)^{2n+1}\left(\frac{3}{2}\right)_n=H'_{2n+1}(0)\end{aligned}\right\} \tag{5.16}$$

6. The Rodrigues Formula

The Rodrigue's formula for H_n(x,y,z) is given by the following relation :

$$H_n(x,y,z)=(-1)^n(x^2+y^2+z^2)^{\frac{n}{2}}\,e^{\frac{x^2y^2z^2}{x^2+y^2+z^2}}\,\frac{d^n}{d\left(\frac{xyz}{\sqrt{x^2+y^2+z^2}}\right)^n}\,e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}} \tag{6.1}$$

a formula of the same nature as Rodrigues's formula for Hermite polynomial of one variable $H_n(x)$.

7. Expansion of Polynomials

Some expansions of Legendre polynomials of one and two variables in a series of Hermite polynomials of three variables are as follows

$$P_n(xyz) = \sum_{k=0}^{\left[\frac{n}{2}\right]} F^{(3)}\left[\begin{matrix} -k, \frac{1}{2}+n-k :: -;-;- : -;-;-; \\ - :: -;-;- : -;-;-; \end{matrix} x^2, y^2, z^2\right] \times \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y,z)}{k!\,(n-2k)!} \quad (7.1)$$

$$P_n(xyz, w) = \sum_{k=0}^{\left[\frac{n}{2}\right]} F^{(3)}\left[\begin{matrix} -k, \frac{1}{2}+n-k :: -;-;- : -;-;-; \\ - :: -;-;- : -;-;-; \end{matrix} \frac{x^2}{1+w}, \frac{y^2}{1+w}, \frac{z^2}{1+w}\right] \times \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y,z)\,(1+w)^k}{k!\,(n-2k)!} \quad (7.2)$$

where

$$P_n(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k} (1+y)^k}{k!\,(n-2k)!}$$

is the Legendre polynomials of two variables defined and studied by M.A. Khan and S. Ahmad [3]

8. Binomial and Trinomial Operator Representations

In a recent paper in 2008, M.A. Khan and A.K. Shukla [2] obtained binomial and trinomial operator representations of certain polynomials. Using their technique we have obtained certain results of binomial and trinomial operator representation type for Hermite polynomials of three variables $H_n(x,y,z)$ by using their Rodrigues formula. Here we need the following results of [2]:

$$(D_x + D_y)^n \{f(x)\, g(y)\} = \sum_{r=0}^{n} \binom{n}{r} D_x^{n-r} f(x)\, D_y^r\, g(y) \quad (8.1)$$

$$(D_x + D_y + D_z)^n \{f(x)\, g(y)\, h(z)\} = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (-1)^{r+s}}{r!\,s!} D_x^{n-r-s} f(x)\, D_y^r\, g(y)\, D_z^s\, h(z) \quad (8.2)$$

and

$$(D_x D_y + D_x D_z + D_y D_z)^n \{f(x)\, g(y)\, h(z)\}$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(-1)^{r+s}}{r!s!}D_x^{n-s}\,f(x)\,D_y^{n-r}\,g(y)\,D_z^{r+s}\,h(z) \tag{8.3}$$

The results obtained are as follows :

Let $\frac{d}{d\left(\frac{xyz}{\sqrt{x^2+y^2+z^2}}\right)}\equiv D_1$ and $\frac{d}{d\left(\frac{uvw}{\sqrt{u^2+v^2+w^2}}\right)}\equiv D_2$, then

$$(D_1+D_2)^n\left\{e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}}.e^{-\frac{u^2v^2w^2}{u^2+v^2+w^2}}\right\}=(-1)^n(x^2+y^2+z^2)^{-\frac{n}{2}}e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}}e^{-\frac{u^2v^2w^2}{u^2+v^2+w^2}}$$

$$\times\sum_{r=0}^{n}\binom{n}{r}\left(\sqrt{\frac{x^2+y^2+z^2}{u^2+v^2+w^2}}\right)^r H_{n-r}(x,y,z)H_r(u,v,w) \tag{8.4}$$

Again let $\frac{d}{d\left(\frac{xyz}{\sqrt{x^2+y^2+z^2}}\right)}\equiv D_1$ $\frac{d}{d\left(\frac{uvw}{\sqrt{u^2+v^2+w^2}}\right)}\equiv D_2$ *and* $\frac{d}{d\left(\frac{fgh}{\sqrt{f^2+g^2+h^2}}\right)}\equiv D_3$, then

$$(D_1+D_2+D_3)^n\left\{e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}}.e^{-\frac{u^2v^2w^2}{u^2+v^2+w^2}}.e^{-\frac{f^2g^2h^2}{f^2+g^2+h^2}}\right\}$$

$$=(-1)^n(x^2+y^2+z^2)^{-\frac{n}{2}}e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}}e^{-\frac{u^2v^2w^2}{u^2+v^2+w^2}}e^{-\frac{f^2g^2h^2}{f^2+g^2+h^2}}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}$$

$$\times\left(\sqrt{\frac{x^2+y^2+z^2}{u^2+v^2+w^2}}\right)^r\left(\sqrt{\frac{x^2+y^2+z^2}{f^2+g^2+h^2}}\right)^s H_{n-r-s}(x,y,z)\,H_r(u,v,w)\,H_s(f,g,h) \tag{8.5}$$

and

$$(D_1D_2+D_1D_3+D_2D_3)^n\left\{e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}}.e^{-\frac{u^2v^2w^2}{u^2+v^2+w^2}}.e^{-\frac{f^2g^2h^2}{f^2+g^2+h^2}}\right\}$$

$$=(x^2+y^2+z^2)^{-\frac{n}{2}}(u^2+v^2+w^2)^{-\frac{n}{2}}e^{-\frac{x^2y^2z^2}{x^2+y^2+z^2}}e^{-\frac{u^2v^2w^2}{u^2+v^2+w^2}}e^{-\frac{f^2g^2h^2}{f^2+g^2+h^2}}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\binom{n}{r}\binom{n-r}{s}$$

$$\times\left(\sqrt{\frac{u^2+v^2+w^2}{f^2+g^2+h^2}}\right)^r\left(\sqrt{\frac{x^2+y^2+z^2}{f^2+g^2+h^2}}\right)^s H_{n-s}(x,y,z)H_{n-r}(u,v,w)\bar{\bar{H}}_{r+s}(f,g,h) \qquad (8.6)$$

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