On Fuzzy BG-Semi Group

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ABSTRACT

In this paper, we introduce the notion of BG-semi group and fuzzy sub BG-semi group of a BG-semi group X and then some related properties are investigated. Characterizations of BG-semi group and fuzzy BG-semi group of a BG-semi group X are given.

Key words: BG-algebra, sub BG-algebra , BG-semi group and fuzzy sub BG-semi group.

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1. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK – algebras and BCI–algebras. It is known that the class of BCK–algebras is a proper subclass of the class of BCI–algebras. J. Neggers and H.S.Kim introduced a new notion, called B–algebra. C.B.Kim and H.S.Kim introduced the notion of the BG–algebra which is a generalization of B–algebra. In this paper, we introduce the concept of BG-semi group and fuzzy sub BG-semi group and discussed some of its properties.

2. PRELIMENARIES

In this section we site the fundamental definitions that will be used in the sequel.

2.1 Definition

A nonempty set X with a constant 0 and a binary operation ' . ' is called a BG–Algebra if it satisfies the following axioms.

1. $x \cdot x = 0$,

2. $x \cdot 0 = x$,

3. $(x * y) * (0 * y) = x$, $\forall$ $x, y \in X$.

2.1 Example

Let $X = \{0,1,2\}$ be the set with the following table.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is a BG – Algebra.

2.2 Definition

Let S be a non empty subset of a BG -algebra X, then S is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

2.3 Definition

A BG-semi group is a nonempty set X with a constant 0 and two binary operation '$*$' and '.' satisfying the following axioms.

i. $(X, *, 0)$ is a BG-algebra.

ii. $(X, .)$ is a semi group.

iii. $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $x * (y \cdot z) = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

2.2 Example

Let $X = \{0,1,2\}$ and define '$*$' and '.' by the following table.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Then $(X, *, .)$ is a BG – semi group.

2.5 Definition

If X is a BG-semi-group, then the relation '$\leq$' by $x \leq y$ if and only if $x * y = 0$.

Remark: we shall write the multiplication $x . y$ by xy.

2.1 Theorem

If X be a BG-semi group then we have,

i. $0x = x0 = 0$

ii. $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ $\forall$ $x, y, z \in X$.

Proof

i. $0x = (x * x) . x = (x . x) * (x . x) = 0.$

$x0 = x . (x * x) = (x . x) * (x . x) = 0.$

ii. $x \leq y$ then $x * y = 0.$

$xz * yz = (x . z) * (y . z) = (x * y) . z = 0 . z = 0.$

Hence $xz \leq yz$.

and

$zx * zy = (z . x) * (z . y) = z . (x * y) = z . 0 = 0.$

Hence $zx \leq zy$

and

$zx * zy = (z . x) * (z . y) = z . (x * y) = z . 0 = 0.$

Hence $zx \leq zy$

Hence $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ $\forall$ $x, y, z \in X$.

2.6 Definition

A non empty subset A of a BG-semi group X is called a left (resp. right) ideal of X if it satisfies following conditions:

i. $0 \in A$,

ii. whenever $x \in X$ and $a \in A$ then $xa \in A$ (resp. $ax \in A$)

iii. For any $x, y \in X$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

2.7 Definition

A both left and right ideal is called two sided ideal (or) simply an ideal.

Note that { 0 } and X are ideals.

2.3 Example

Let X = { 0,1,2,3 } and define '$*$' and '.' by the following table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Clearly $(X, *, .)$ is a BG – semi group.

If $A = \{0, 1\}$, then A is an ideal of a BG-semi group X.

If $A = \{0, 2\}$, then A is an ideal of a BG-semi group X.

2.8 Definition

For any $x, y \in X$, denote $x \wedge y = x * (x * y)$.

Clearly $x \wedge x = x * (x * x) = x * 0 = x$.

$x \wedge 0 = x * (x * 0) = x * x = 0$.

$0 \wedge x = 0 * (0 * x) = (x * x) * (0 * x) = x$.

2.9 Definition

Let X and Y be any two BG-semi groups. A mapping $f : X \to Y$ is called a BG-semi group homomorphism (briefly homomorphism) if it satisfies the following conditions.

i. $f(x * y) = f(x) * f(y) \quad \forall x, y \in X$.

ii. $f(xy) = f(x) f(y) \quad \forall x, y \in X$.

2.10 Definition

Let $f : X \to Y$ be a homomorphism of BG-semi groups, then the set $\{x \in X / f(x) = 0\}$ is called the kernel of f and it is denoted by **Ker f.**

2.2 Theorem

Let $f : X \to Y$ be a homomorphism of BG-semi groups, Then

i. $f(0) = 0$

ii. $x \leq y$ implies $f(x) \leq f(y)$

iii. $f(x \wedge y) = f(x) \wedge f(y)$

Proof

i. $f(0) = f(x * x) = f(x) * f(x) = 0$.

ii. $x \leq y$ implies $x . y = 0$

$f(x).f(y) = f(x.y) = f(0) = 0.$

Hence $f(x) \leq f(y)$.

iii. $f(x \wedge y) = f(x.(x.y)) = f(x).f(x.y) = f(x).(f(x).f(y))$

$= f(x) \wedge f(y).$

Hence $f(x \wedge y) = f(x) \wedge f(y)$

2.3 Theorem

Let $f: X \rightarrow Y$ be a homomorphism of BG-semi groups and $J = f^{-1}(0) = \{0\}$ then

$f(x) \leq f(y)$ implies $x \leq y$.

Proof

If $f(x) \leq f(y)$ then

$0 = f(x).f(y) = f(x.y).$

$x.y \in J$ and $x.y = 0.$

Hence $x \leq y$.

2.4 Theorem

Let $f: X \rightarrow Y$ be a homomorphism of BG-semi groups then Ker f is an ideal of X.

Proof

i. Clearly $0 \in$ Ker f.

ii. Let $x \in X$ and $a \in$ Ker f then $f(a) = 0$

$f(ax) = f(a)f(x) = 0 f(x) = 0$, then $ax \in$ Ker f.

$f(xa) = f(x)f(a) = f(x)0 = 0$, then $xa \in$ Ker f.

i. For any $x, y \in X$, $x.y \in$ Ker f and $y \in$ Ker f, then $f(x.y) = 0$ and $f(y) = 0.$

$0 = f(x.y) = f(x).f(y) = f(x).0 = f(x).$

$f(x) = 0$ and $x \in$ Ker f.

Hence Ker f is an ideal of X.

3. FUZZY SUB BG-SEMI GROUP

In this section, we introduce the notion of fuzzy sub BG-semi group of a BG-semi group X and discussed some of its properties.

3.1 Definition

Let X be a BG-semi group. A fuzzy subset μ in X is called a fuzzy sub BG-semi group of X if it satisfies the following conditions.

i. $\mu(x * y) \geq \min\{\mu(x), \ (y)\}$,

ii. $\mu(xy) \geq \min\{\mu(x), \ \mu(y)\}$, for all x and y $\in$ X

3.1 Example

Let X = { 0,1,2 } and define '$*$' and '.' by the following table.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Clearly $(X, *, .)$ is a BG – semi group.

Define a fuzzy subset $\mu : X \to [0, 1]$ by,

$$\mu(x) = \begin{cases} 0.4 & \text{for all } x \neq 0 \\ 0.7 & \text{for } x = 0. \end{cases}$$

Clearly μ is a fuzzy sub BG-semi group of X.

3.2 Definition

Let μ be a fuzzy subset of S. For t $\in$ [0 , 1], the level subsets of μ is the set,

$\mu_t = \{ x \in S : \mu(x) \geq t \}$.

3.1 Theorem

Let μ be a fuzzy sub BG-semi group of X. Then for t $\in$ [0,1] such that t $\leq$ (0), μ_t is either empty or a sub BG-semi group of X.

Proof

Let $\mu_t \neq \phi$ and for all x, y in μ_t, we have,

$\mu(x) \geq t$; $\mu(y) \geq t$.

Now $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

$\mu(x * y) \geq \min\{t, t\}$

$\mu(x * y) \geq t$

$x * y \in \mu_t$

Hence μ_t is a sub BG- semi group of x.

3.2 Theorem

Let X be a BG-semi group and μ be a fuzzy subset of X such that μ_t is a sub BG-semi group of X. For $t \in [0,1]$ such that $t \leq \mu(0), \mu$ is a fuzzy sub BG-semi group of X .

Proof

Let x, y in X and $\mu(x) = t_1$ and $\mu(y) = t_2$.

Suppose $t_1 > t_2$, then $x, y \in \mu_{t2}$.

As μ_{t2} is a sub BG-semi group of X , $x . y \in \mu_{t2}$.

Hence, $\mu(x . y) \geq t_2 = \min\{t_1, t_2\}$

$\geq \min\{\mu\{x), \mu(y)\}$

That is , $\mu(x . y) \geq \min\{\mu(x), (y)\}$.

and if $x, y \in \mu_{t2}$ and $_{t2}$ is a sub BG-semi group of X , then $xy \in \mu_{t2}$.

Hence, $\mu(xy) \geq t_2 = \min\{t_1, t_2\}$

$\geq \min\{\mu(x), \mu(y)\}$

That is , $\mu(xy) \geq \min\{\mu(x), (y)\}$.

Hence μ is a fuzzy sub BG-semi group of X .

3.3 Definition

Let X be a BG-semi group and μ be a fuzzy sub BG-semi group of X . Then the sub BG-semi groups μ_t for $t \in [0,1]$ and $t \leq \mu(0)$, are called level sub BG-semi groups of X.

3.3 Theorem

Any sub BG-semi group H of a BG-semi group X can be realized as a level sub BG-semi group of some fuzzy sub BG-semi group of X.

Proof

Let μ be a fuzzy subset and $x \in X$.

$\mu(x) = t$ if $x \in H$

0 if $x \notin H$, where $t \in (0,1]$.

We shall prove that μ is a fuzzy sub BG-semi group of X.

Let $x, y \in X$.

i. Suppose $x, y \in H$, then $x * y \in H$ and $xy \in H$

$\mu(x) = t$, $\mu(y) = t$, $\mu(x * y) = t$ and $\mu(xy) = t$.

Hence $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ and

$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

ii. Suppose $x \in H$ and $y \notin H$, then $x * y \notin H$.

$\mu(x) = t$, $\mu(y) = 0$, $\mu(x * y) = 0$ and $\mu(xy) = 0$.

Hence $\mu(x * y) \geq \min\{(x), (y)\}$ and

$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

iii. Suppose $x, y \notin H$, then $x * y \in H$ or $x * y \notin H$ and $xy \in H$ or $xy \notin H$.

$\mu(x) = 0$, $\mu(y) = 0$, then $\mu(x * y) = t$ or 0 and $\mu(xy) = t$ or 0.

Hence $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ and

$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

Thus in all cases, μ is a fuzzy sub BG-semi group of X

For this fuzzy sub BG-semi group, $\mu_t = H$.

3.4 Theorem

Let μ be fuzzy sub BG-semi group of X. Two level subgroups μ_{t1}, μ_{t2}, for, $t_1, t_2 \in [0,1]$ and $t \leq \mu(e)$ with $t_1 < t_2$ of are equal if and only if there is no x in X such that $t_1 \leq \mu(x) < t_2$.

Proof:

Let $\mu_{t1} = \mu_{t2}$.

Suppose there exists a $x \in X$ such that $t_1 < \mu(x) < t_2$ then $\mu_{t2} \subseteq \mu_{t1}$.

Then $x \in \mu_{t1}$, but $x \notin \mu_{t2}$, which contradicts the assumption that, $\mu_{t1} = \mu_{t2}$.

Hence there is no $x \in X$ such that $t_1 < \mu(x) < t_2$.

Conversely, let there is no $x \in X$ such that $t_1 < \mu(x) < t_2$.

As $t_1 < t_2$ imply that $\mu_{t2} \subseteq \mu_{t1}$ and there is no $x \in X$ such that $t_1 < \mu(x) < t_2$.

Then $\mu_{t1} \subseteq \mu_{t2}$.

Hence $\mu_{t1} = \mu_{t2}$.

3.5 Theorem

A fuzzy subset μ of a BG-semi group X is a fuzzy sub BG-semi group of X if and only if the level subsets μ_t, $t \in$ Image μ, are sub BG-semi groups of X.

Proof : It is clear.

REMARK

As a consequence of the above Theorem, the level sub BG-semi groups of a fuzzy sub BG-semi group of a BG-semi group X form a chain. Since $\mu(x) \leq \mu(0)$ for all x in X, therefore μ_{t0}, where $\mu(x) = t_0$ is the greatest and we have the chain :

$\{0\} = \mu_{t0} \subset \mu_{t1} \subset \mu_{t2} \subset \ldots \subset \mu_{tn} = X$, where $t_0 > t_1 > t_2 > \ldots > t_n$.

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