Distance-g domination in Cayley graphs

T. Tamizh Chelvam [a]

[a] *Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India*
E-mail: tamche_59@yahoo.co.in

L. Barani Kumar [b]

[b] *Department of Mathematics, Adhiparasakthi Engineering College, Melmaruvathur-603 319, Tamil Nadu, India*
E-mail: barani_apec@yahoo.com

Sivagnanam Mutharasu [c]

[c] *Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India*
Email: skannanmunna@yahoo.com

Abstract

A Cayley graph is a graph constructed out of a group Γ and its generating set A. In this paper we attempt to find distance$-g$ dominating sets in Cayley graphs constructed out of $\mathbb{Z}_n$. Actually we find the value of distance$-g$ domination number for $Cay(\mathbb{Z}_n, A)$ where A is a generating set of $\mathbb{Z}_n$. Further we have proved that $Cay(\mathbb{Z}_n, A)$ is distance$-g$ excellent. We have also shown that $Cay(\mathbb{Z}_n, A)$ is distance$-g$ $2-$excellent if and only if $n = t(g|A| + 1) + 1$ for some positive integer t. Also we proved that some Cayley graphs are distance$-g$ $2-$restricted.

Key words: Cayley graphs, Distance-g dominating sets, Distance-g excellent, Distance-g restricted.
AMS Subject Classification: 05C.
Received: November 30, 2008.

1 Introduction

Let Γ be a finite group with e as the identity. A generating set of the group Γ is a subset A such that every element of Γ is a product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The Cayley graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, y)_a | \ x, y \in V(G)$, there exists $a \in A$ such that $y = xa\}$ and it is denoted by $Cay(\Gamma, A)$. The exclusion of e from A eliminates the possibility of loops in the graph. The inclusion of the inverse in A for every element of A means that an edge is in the graph regardless of which end vertex is considered. Note that G is connected and $|A|$ is the degree of $Cay(\Gamma, A)$ [5]. Also the Cayley graphs constructed out of finite cyclic groups are called circulant graphs.

The concept of domination for circulant graphs has been studied by various authors and one can refer to [2], [9], [7], [10]. I.J. Dejter and O. Serra [2] obtained efficient dominating sets for Cayley graphs constructed on permutation groups. The efficient domination number for vertex transitive graphs have been obtained by J.Huang and J-M. Xu [7], where as efficient domination in circulant graphs with two chord lengths are studied by N. Obradovič J. Peters and G. Ružić [10]. The existence of independent perfect domination sets in Cayley graphs was studied by J.Lee [9]. Tamizh Chelvam and Rani [11] obtained the domination number for certain Cayley graphs constructed on $\mathbb{Z}_n$ with respect to a particular generating set of $\mathbb{Z}_n$. Further Tamizh Chelvam and Rani [12] obtained the independent domination number for Cayley graphs on $\mathbb{Z}_n$.

Suppose $G = (V, E)$ is a connected graph, the open neighbourhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of vertices adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. Let $u, v \in V(G)$. Then $d(u, v)$ is the length of a shortest $uv-$path. For any $v \in V(G), N_{\leq g}(v) = \{u \in V(G) : d(u, v) \leq g\}$ and $N_{\leq g}[v] = N_{\leq g}(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood $N_{\leq g}(S)$ is defined to be $\cup_{v \in S} N_{\leq g}(v)$, and the closed neighbourhood of S is $N_{\leq g}[S] = N_{\leq g}(S) \cup S$. A set $S \subseteq V$ is called a dominating set if every vertex $v \in V$ is either an element of S or adjacent to an element of S[1]. The domination number $\gamma(G)$ of G is the minimum cardinality among all the dominating sets in G[1] and the corresponding dominating set is called a γ-set. A set $S \subseteq V$, is called a distance$-g$ dominating set if $N_{\leq g}[S] = V(G)$. The distance$-g$ domination number $\gamma_{\leq g}(G)$ is the minimum cardinality among all the distance$-g$ dominating sets in G[1] and the corresponding distance$-g$ dominating set is called a $\gamma_{\leq g}$-set.

A dominating set $S \subseteq V(G)$ is said to be efficient dominating set if $|N[v] \cap S| = 1$ for all $v \in V(G)$. A distance$-g$ dominating set $S \subseteq V(G)$ is said to be distance$-g$ efficient dominating set if for all $v \in V(G)$, $|N_{\leq g}[v] \cap S| = 1$. Note that all the efficient dominating sets have the same cardinality $\gamma(G)$ and all the distance$-g$ efficient dominating set have the same cardinality $\gamma_{\leq g}(G)$. A graph G is said to be distance$-g$ excellent(or distance$-g$ restricted) if each vertex u of G is contained(or not contained) in some $\gamma_{\leq g}$-set of G. The graph G is said to be distance$-g$ $k-$excellent(or distance$-g$ $k-$restricted), if every subset $S \subseteq V$ with $|S| = k$ is contained(or not contained) in some $\gamma_{\leq g}$-set of G. Throughout this paper, n is a fixed positive integer, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and $G = Cay(\mathbb{Z}_n, A)$, where A is a generating set. Unless otherwise specified A stands for the set $\{1, n-1, 2, n-2, \ldots, k, n-k\}$ where $1 \leq k \leq \frac{n-1}{2}$. Hereafter $+$ stands for modulo n addition in $\mathbb{Z}_n$.

2 Distance$-g$ Domination in $G = Cay(\mathbb{Z}_n, A)$

In this section, the value of distance$-g$ domination number for certain Cayley graphs is obtained. Also the existence of distance$-g$ 2-excellent Cayley graphs has been discussed.

Theorem 2.1 *Let n, k be integers such that $1 \leq k \leq \frac{n-1}{2}$ and $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$. Then $\gamma_{\leq g}(G) = \lceil \frac{n}{g|A|+1} \rceil$.*

Proof: Let $\ell = \lceil \frac{n}{g|A|+1} \rceil = \lceil \frac{n}{2gk+1} \rceil$. Then $n = (\ell - 1)(1 + 2gk) + h$ for some h with $1 \leq h \leq 2gk + 1$. Note that, for any $v \in V(G)$, $N_1[v] = \{v, v+1, v+2, \ldots, v+k, v+(n-1), v+(n-2), \ldots, v+(n-k)\}$. Hence by using the property of vertex transitivity one can write $N_{\leq g}[v] = \{v, v+1, v+2, \ldots, v+gk, v+(n-1), v+(n-2), \ldots, v+(n-gk)\}$. Hence $\gamma_{\leq g}(G) \geq \lceil \frac{n}{1+2gk} \rceil$.
Consider the set $D = \{0, (2gk+1), 2(2gk+1), 3(2gk+1), \ldots, (\ell-1)(2gk+1)\}$.
Claim. D is a distance$-g$ dominating set of G.
It is always true that $N_{\leq g}[D] = \bigcup_{i=0}^{\ell-1} N_{\leq g}[(2gk+1)i] \subseteq V(G)$ and so it is enough to prove that $V(G) \subseteq N_{\leq g}[D]$. Let $c \in V(G)$. By division algorithm, one can write $c = (2gk+1)i + j$ for some i and j satisfying $0 \leq i \leq \ell - 1$ and $0 \leq j \leq 2gk$. We have the following cases:
Case(i). Suppose $0 \leq j \leq gk$ and $0 \leq i \leq \ell - 1$, then it is easy to see that $c \in N_{\leq g}[(2gk+1)i] \subseteq N_{\leq g}[D]$.
Case(ii). Suppose $gk + 1 \leq j \leq 2gk$ and $0 \leq i \leq \ell - 2$. In this case $c = (2gk+1)i + j = (2gk+1)(i+1) + (j - (1+2gk))$. Hence $c + ((1+2gk) - j) = (2gk+1)(i+1)$ where $1 \leq i+1 \leq \ell - 1$. Since $gk + 1 \leq j \leq 2gk$, we have $1 \leq (1+2gk) - j \leq gk$. Hence $c \in N_{\leq g}[(2gk+1)(i+1)] \subseteq N_{\leq g}[D]$.
Case(iii). Assume that $gk + 1 \leq j \leq 2gk$ and $i = \ell - 1$. In this case $0 = (2gk+1)(\ell-1) + j + (h - j) = c + (h - j)$. By the assumption on j and h, we have $gk + 1 \leq j \leq h \leq 2gk$. Hence $0 \leq h - j \leq gk - 1$. This implies that $c \in N_{\leq g}[0] \subseteq N_{\leq g}[D]$. Now we have $V(G) \subseteq N_{\leq g}[D]$ and so $\gamma_{\leq g}(G) \leq |D| = \ell = \lceil \frac{n}{g|A|+1} \rceil$.

Remark 2.2 $\gamma_{\leq g}(Cay(\mathbb{Z}_n, A))$ depends upon A, the generating set of $\mathbb{Z}_n$. For example, let $g = 1$ and consider $Cay(\mathbb{Z}_{10}, A)$ with $|A| = 4$. It is obvious that $\{1,2,8,9\}$ and $\{1,4,6,9\}$ are generating sets. When $A = \{1,2,8,9\}$, $\gamma_{\leq g}(Cay(\mathbb{Z}_{10}, A)) = 2$ and when $A = \{1,4,6,9\}$, $\gamma_{\leq g}(Cay(\mathbb{Z}_{10}, A)) = 3$.

In the following **Lemma**, we identify another $\gamma_{\leq g}$-set for $Cay(\mathbb{Z}_n, A)$ apart from the one identified in the proof of Theorem 2.1.

Lemma 2.3 *Let n, k be integers such that $1 \leq k \leq \frac{n-1}{2}$ and $\ell = \lceil \frac{n}{2gk+1} \rceil$. Then for fixed k, if $n = (2gk+1)(\ell-1)+h, 1 \leq h \leq 2gk$ then $D = \{0, h, h+(2gk+1), \dots, h+(\ell-2)(2gk+1)\}$ is a $\gamma_{\leq g}$-set for G.*

Proof: Note that any $v \in V(G)$ could be written as $v = h+(2gk+1)i+r$ for some i and r with $0 \leq i \leq \ell-1$ and $0 \leq r \leq 2gk$.
The following cases arise:
Case (i). Suppose $v = h+(2gk+1)i+r$, $0 \leq i \leq \ell-2$ and $0 \leq r \leq gk$ then $v \in N_{\leq g}[h+(2gk+1)i] \subseteq N_{\leq g}[D]$. Further when $i = \ell-1$ and $0 \leq r \leq gk, v \in N_{\leq g}[0] \subseteq N_{\leq g}[D]$.
Case (ii). Suppose $v = h+(2gk+1)i+r$, $0 \leq i \leq \ell-1$ and $gk+1 \leq r \leq 2gk$.
Subcase (i). When $0 \leq i \leq \ell-3$, we have $v = h+(2gk+1)(i+1)+(r-(2gk+1))$ and so $v+((2gk+1)-r) = h+(2gk+1)(i+1)$ where $1 \leq (2gk+1)-r \leq gk$ and $1 \leq i+1 \leq \ell-2$. Hence $v \in N_{\leq g}[h+(2gk+1)(i+1)] \subseteq N_{\leq g}[D]$.
Subcase (ii). When $i = \ell-2$, we have $v = h+(2gk+1)(\ell-1)+(r-(2gk+1)) = 0+(r-(2gk+1))$ and so $v+((2gk+1)-r) = 0$ where $1 \leq (2gk+1)-r \leq gk$. Hence $v \in N_{\leq g}[0] \subseteq N_{\leq g}[D]$.
Subcase (iii). Suppose $i = \ell-1$, we have $v = h+(2gk+1)(\ell-1)+r = r$, In this case $v \in N_{\leq g}[\{h, h+(2gk+1)\}] \subseteq N_{\leq g}[D]$. In all the cases $v \in N_{\leq g}[D]$ and hence by Theorem 2.1, D is a $\gamma_{\leq g}$-set.

Lemma 2.4 *Assume that $A_1 = \{1, n-1\}$. Suppose D is a $\gamma_{\leq g}$-set for $G_1 = Cay(\mathbb{Z}_n, A_1)$, then D is a distance$-g$ dominating set for $G_2 = Cay(\mathbb{Z}_n, A_2)$ with $A_2 = \{1, n-1, 2, n-2, \dots\}$ and $|A_2| > 2$.*

Proof: Since $A_1 \subset A_2, G_1 = Cay(\mathbb{Z}_n, A_1)$ is a spanning subgraph of $G_2 = Cay(\mathbb{Z}_n, A_2)$, the result follows.

Remark 2.5 If D is a distance$-g$ dominating set, then by the property of vertex transitivity in Cayley graphs, $D+v$ is also a distance$-g$ dominating set for all $v \in V(G)$. Thus Cayley graph $G = Cay(\mathbb{Z}_n, A)$ is distance$-g$ excellent.

Lemma 2.6 *Let n, k be integers such that $1 \leq k \leq \frac{n-1}{2}$ and $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \dots, k, n-k\}$. If $n = (2gk+1)t+1$ for some positive integer t, then G is distance$-g$ $2-$excellent.*

Proof: Let $\ell = \lceil \frac{n}{g|A|+1} \rceil$. By Theorem 2.1, $\gamma_{\leq g}(G) = \ell = t+1$ and $D = \{0, (2gk+1), 2(2gk+1), 3(2gk+1), \ldots, (\ell-1)(2gk+1)\}$ is $\gamma_{\leq g}-$ set. Since the Cayley graphs have the property of vertex transitivity, it is enough to prove that for any given $d \in V(G)$, $d \neq 0$, there exists a $\gamma_{\leq g}-$set D_1 such that $\{0, d\} \subseteq D_1$.

Let $d (\neq 0) \in V(G)$. If $d \in D$ then nothing to prove. Otherwise d lies between $i(2gk+1)$ and $(i+1)(2gk+1)$ for some $0 \leq i \leq \ell-2$. Since $(\ell-1)(1+2gk)+1 \equiv 0 \ (mod\ n)$, there exists no element between $(\ell-1)(1+2gk)$ and 0. Having i fixed, consider the set $D_1 = \{0, 1+2gk, 2(1+2gk), \ldots, i(1+2gk), d, (i+1)(1+2gk)+1, \ldots, (\ell-2)(1+2gk)+1\}$ Note that $|D_1| = \ell$. Let $v \in V(G)$. If $v \in D_1$, then nothing to prove. Otherwise we have the following cases:

Case (i). If $v \in \{1, 2, \ldots, i(1+2gk)-1\}$. Then $v = r(1+2gk)+j$ for some r and j with $0 \leq r \leq i-1$ and $1 \leq j \leq 2gk$.

Subcase (i). If $0 \leq r \leq i-1$ and $1 \leq j \leq gk$ then $v \in N_{\leq g}[r(1+2gk)] \subseteq N_{\leq g}[D_1]$.

Subcase (ii). Suppose $0 \leq r \leq i-1$ and $gk+1 \leq j \leq 2gk$. In this case $v = (2gk+1)r+j = (2gk+1)(r+1)+(j-(1+2gk))$. Hence $v+((1+2gk)-j) = (2gk+1)(r+1)$ where $1 \leq r+1 \leq i$. Since $gk+1 \leq j \leq 2gk$, we have $1 \leq (1+2gk)-j \leq gk$. Hence $v \in N_{\leq g}[(2gk+1)(r+1)] \subseteq N_{\leq g}[D_1]$.

Case (ii). If $v \in \{i(1+2gk)+1, \ldots, (i+1)(1+2gk)\}$.

Subcase (i). Suppose $v = i(1+2gk)+j$ for some j with $1 \leq j \leq gk$. Then $v \in N_{\leq g}[i(1+2gk)] \subseteq N_{\leq g}[D_1]$.

Subcase (ii). Suppose $v = i(1+2gk)+gk+1$. By the definition of d, $d = i(2gk+1)+x$ for some x with $1 \leq x \leq 2gk$.

If $1 \leq x \leq gk$ then $1 \leq (1+gk)-x \leq gk$ and $d+((1+gk)-x) = v$. Hence $v \in N_{\leq g}[d] \subseteq N_{\leq g}[D_1]$.

Otherwise $gk+2 \leq x \leq 2gk$ and so $1 \leq x-(1+gk) \leq gk-1$. In this case $v+(x-(1+gk)) = d$ and so $v \in N_{\leq g}[d] \subseteq N_{\leq g}[D_1]$.

Subcase (iii). Suppose $v = i(1+2gk)+j$ for some j with $gk+2 \leq j \leq 2gk+1$ In this case $v = (2gk+1)i+j = (2gk+1)(i+1)+1+(j-(2gk+2))$. Hence $v+((2+2gk)-j) = (2gk+1)(i+1)+1$. Since $gk+2 \leq j \leq 2gk+1$, we have $1 \leq (2+2gk)-j \leq gk$. Hence $v \in N_{\leq g}[(2gk+1)(i+1)+1] \subseteq N_{\leq g}[D_1]$.

Case (iii). If $v \in \{(i+1)(1+2gk)+2, (i+1)(1+2gk)+3, \ldots, (\ell-1)(1+2gk)(=n-1)\}$. Then $v = r(1+2gk)+1+j$ for some r and j with $i+1 \leq r \leq \ell-2$ and $1 \leq j \leq 2gk$.

Subcase (i). Suppose $i+1 \leq r \leq \ell-2$ and $1 \leq j \leq gk$. Then $v \in N_{\leq g}[r(1+2gk)+1] \subseteq N_{\leq g}[D_1]$.

Subcase (ii). Suppose $i+1 \leq r \leq \ell-3$ and $gk+1 \leq j \leq 2gk$. In this case $v = (2gk+1)r+1+j = (2gk+1)(r+1)+1+(j-(1+2gk))$. Hence $v+((1+2gk)-j) = (2gk+1)(r+1)+1$ where $i+2 \leq r+1 \leq \ell-2$. Since $gk+1 \leq j \leq 2gk$, we have $1 \leq (1+2gk)-j \leq gk$. Hence $v \in N_{\leq g}[(2gk+1)(r+1)+1] \subseteq N_{\leq g}[D_1]$.

Subcase (iii). Suppose $r = \ell-2$ and $gk+1 \leq j \leq 2gk$. In this case $v =$

$(2gk+1)(\ell-2)+1+j = (2gk+1)(\ell-1)+1+(j-(1+2gk))$. Hence $v+((1+2gk)-j) = (2gk+1)(\ell-1)+1 = 0$. Since $gk+1 \leq j \leq 2gk$, we have $1 \leq (1+2gk)-j \leq gk$. Hence $v \in N_{\leq g}[0] \subseteq N_{\leq g}[D_1]$.
Hence D_1 is a $\gamma_{\leq g}$−set which contains the set $\{0, d\}$ and hence G is distance−g 2−excellent.

Theorem 2.7 *Let n, k be integers such that $1 \leq k \leq \frac{n-1}{2}$ and $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$. If $n = (2gk+1)t + j$ for some integer $t > 0$ and $1 \leq j \leq 2gk+1$, then G is distance−g 2-excellent if and only if $j = 1$.*

Proof: Suppose $j = 1$. Then $n = t(2gk+1)+1$ and hence by Lemma 2.6, G is distance−g 2− excellent.
Conversely let G be distance−g 2-excellent. Suppose $j \neq 1$. Then $n = t(2gk+1)+j$, for some $j, 1 < j \leq 2gk+1$. Consider the two vertices $gk+1$ and $gk+2$. Since G is distance−g 2-excellent, there exists a $\gamma_{\leq g}$−set D such that $\{gk+1, gk+2\} \subseteq D$. Then by Theorem 2.1, $|D| = \ell = \lceil \frac{n}{2gk+1} \rceil = t+1$. As in the proof of Theorem 2.1, $N_{\leq g}[gk+1]$ contains all elements between 1 and $2gk+1$ and hence $N_{\leq g}[gk+2]$ contains elements between 2 and $2gk+2$. Therefore $N_{\leq g}[\{gk+1, gk+2\}]$ contains exactly $2gk+2$ vertices of G. Since $N_{\leq g}[v] = 2gk+1$, for all $v \in V(G)$, $N_{\leq g}[D]$ can contains at most $(\ell-2)(2gk+1)+2gk+2$ vertices of G, whereas $(\ell-2)(2gk+1)+2gk+2 = (\ell-1)(2gk+1)+1 = t(2gk+1)+1 < n$, which is a contradiction to D is a distance−g dominating set. Hence $j = 1$.

Lemma 2.8 *Let n, k be integers such that $1 \leq k \leq \frac{n-1}{2}$ and $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$. If $n = (2gk+1)t+1$ for some positive integer t, then G is distance−g 2−restricted.*

Proof: By Theorem 2.1, $\gamma_{\leq g}(G) = \lceil \frac{n}{2gk+1} \rceil$ and $D = \{0, 1+2gk, 2(1+2gk), \ldots, (\ell-1)(1+2gk)\}$ is a $\gamma_{\leq g}$−set. Let $x, y \in V(G)$ and assume $x < y$. In order to find a $\gamma_{\leq g}$−set D of G such that $x, y \notin D$, it is enough to find a $\gamma_{\leq g}$−set D_1 such that $0, d \notin D_1$ for any $0 \neq d \in V(G)$. Let $d (\neq 0) \in V(G)$. Then by Lemma 2.6, there exists a $\gamma_{\leq g}$−set $D_1 = \{0, 1+2gk, 2(1+2gk), \ldots, i(1+2gk), d, (i+1)(1+2gk)+1, \ldots (\ell-2)(1+2gk)+1\}$ such that $0, d \in D_1$. Also there are at least two elements in between $i(1+2gk)$ and $(i+1)(1+2gk)$ and d is the only element of D_1 lies in between $i(1+2gk)$ and $(i+1)(1+2gk)$. Note that $n-1, 1 \notin D_1$ and so $0 \notin D_1+1$ and $0 \notin D_1+(n-1)$.
If $d-1 \in D_1$, then $d-1 = i(1+2gk)$. From this $d+1 \notin D_1$ and hence $d \notin D_1+(n-1)$. Hence by Remark 2.5, D_1+n-1 is a $\gamma_{\leq g}$−set and $0, d \notin D_1+(n-1)$. Otherwise $d-1 \notin D_1$ and in this case D_1+1 is a $\gamma_{\leq g}$−set such that $0, d \notin D_1+1$.

In the following theorem, we obtain a necessary and sufficient condition for the existence of distance$-g$ efficient domination sets for $Cay(\mathbb{Z}_n, A)$.

Theorem 2.9 *Let n, k be integers such that $1 \leq k \leq \frac{n-1}{2}$ and $G = Cay(\mathbb{Z}_n, A)$ where $A = \{1, n-1, 2, n-2, \ldots, k, n-k\}$. Let $\ell = \lceil \frac{n}{2gk+1} \rceil$. Then G has a distance$-g$ efficient dominating set if and only if $n = \ell(2gk+1)$. In this case the distance$-g$ efficient domination number is $\frac{n}{2gk+1}$.*

Proof: By Theorem 2.1, we have $\gamma_{\leq g}(G) = \lceil \frac{n}{2gk+1} \rceil$. Assume that D is a distance$-g$ efficient dominating set of G. Then $|D| = \ell$ and $|N_{\leq g}(v) \cap N_{\leq g}(u)| = \emptyset$ for any two distinct vertices $u, v \in D$. Also $|N_{\leq g}[v]| = 1 + 2gk$ for all $v \in V(G)$. Hence $n = \ell(1 + 2gk)$.
Conversely suppose $n = \ell(2gk+1)$. By Theorem 2.1, $D = \{0, (2gk+1), 2(2gk+1), 3(2gk+1), \ldots, (\ell-1)(2gk+1)\}$ is a distance$-g$ dominating set and $|D| = \ell$. Since $|N_{\leq g}[v]| = 2gk+1$ for all $v \in V(G)$, $n = \ell(2gk+1)$ and $|D| = \ell$, one can conclude that D is a distance$-g$ efficient dominating set in G.

Acknowledgement

The work reported here is supported by the Project SR/S4/MS:328/06, awarded to the first and third authors by the Department of Science and Technology, Government of India, New Delhi

References

[1] E. J. Cockayne and S. T. Hedetniemi, Disjoint independent dominating sets in graphs, *Discrete Math.* **15**(1976), 213-222.

[2] I.J. Dejter and O.Serra, Efficient dominating sets in Cayley graphs, *Discrete Appl. Math.*, **129**(2003), 319-328.

[3] D.L. Grinstead and P.J. Slater, On minimum dominating sets with minimum intersection, *Discrete Math.* **86**(1990), 239-254.

[4] F. Harary and T.W. Haynes, Double domination in graphs, *Ars Combin.* **55**(2000), 201-213.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[6] T. W. Haynes and M. A. Henning, Trees with two minimum independent dominating sets, *Discrete Math.* **304**(2005), 69-78.

[7] Jia Huang, Jun-Ming Xu, The bondage numbers and efficient dominations of vertex-transitive graphs, *Discrete Mathematics* **308**(2008), 571-582.

[8] S. Lakshmivarahan and S. K. Dhall,Ring, torus, hypercube architectures algorithms for parallel computing, *Parallel Computing*, **25**(1999), 1877-1906.

[9] J. Lee, Independent perfect domination sets in Cayley graphs, *J. Graph Theory* **37(4)**(2001), 231-219.

[10] N. Obradović, J. Peters and Goran Ružić, Efficient domination in circulant graphs with two chord lengths, *Information Processing Letters* **102**(2007), 253-258.

[11] T. Tamizh Chelvam and I. Rani, Dominating sets in Cayley graphs on $\mathbb{Z}_n$, *Tamkang Journal of Mathematics* **37**, No.4(2007), 341-345.

[12] T. Tamizh Chelvam and I. Rani, Independent domination number for Cayley graphs on $\mathbb{Z}_n$, *The Journal of Combinatorial Mathematics and Combinatorial Computing*, **69**(2009), 251-255.