Normal Fuzzy Biclosure Spaces

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Abstract

The purpose of this paper is to introduce the concept of normal fuzzy biclosure spaces and investigate some of their characterizations.

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1. Introduction

Closure spaces were introduced by E.Čech [3]. The notion of closure system and closure operators are very useful tools in several areas of classical mathematics. They play an important role in topological spaces , Boolean algebra , convex sets etc. This led several authors to investigate the closure operator in the frame work of fuzzy set theory. Gerla et al. [1] studied fuzzy closure operator and fuzzy closure system as extension of closure operator and closure system respectively . Fuzzy closure spaces were first studied by A.S. Mashhour and M.H Ghanim [4,5]. Recently , Chawalit Boonpok introduced the notion of biclosure spaces [2]. Such spaces are equipped with two arbitrary fuzzy closure operators. He extended some of the standard results of separation axioms in closure space to biclosure space. Thereafter a large number of papers have been written to generalize the concept of closure space to biclosure space.

In this paper we study the concept of normal fuzzy biclosure spaces and study some of their properties.

2. Preliminaries

In order to make this paper self contained , we briefly recall certain definitions and results. In this paper , let X be an arbitrary nonempty set , I = [0,1] and I^X be a family of all fuzzy sets of X. For a fuzzy set A of X , cl(A) , int(A) and 1- A will denote the closure of A , the interior of A and the complement of A respectively whereas the constant fuzzy sets taking on the values 0 and 1 on X are denoted by 0_X and 1_X respectively.

Definition 2.1 [5 ,7]

A function $u : I^X \rightarrow I^X$ defined on the family I^X of all fuzzy sets of X is called a fuzzy closure operator on X and the pair (X , u) is called fuzzy closure space , if the following conditions are satisfied

1) $u\,\phi = \phi$

2) $A \leq u\ (A)$ for all A $\in I^X$.

3) $u(A \vee B) = u(A) \vee u(B)$ for all $A, B \in I^X$.

Definition 2.2 [4]

A fuzzy subset A of a fuzzy closure space (fcs) (X,u) is said to be fuzzy closed , if $u\,A = A$ and it is fuzzy open if its complement X - A is fuzzy closed.

The empty set and the whole set are both fuzzy open and fuzzy closed.

Definition 2.3 [4]

A fuzzy closure space (Y, v) is said to be a fuzzy subspace of (X,u) if

Y $\leq$ X and $v\ A = u\ A \wedge Y$ for each fuzzy subset $A \leq$ Y. If Y is fuzzy closed in (X , u) , then the fuzzy subspace (Y, v) of (X, u) is also said to be fuzzy closed

Definition 2.4 [6]

Let (X, u) and (Y, v) be fuzzy closure spaces. A map $f : (X,u) \rightarrow (Y,v)$ is said to be fuzzy continuous if $f(uA) \leq vf(A)$ for every fuzzy subset $A \leq$ X. In other words a map $f : (X,u) \rightarrow (Y,v)$ is fuzzy continuous if and only if $uf^{-1}(B) \leq f^{-1}v(B)$ for every fuzzy subset B $\leq$ Y.

Clearly, if map $f:(X,u)\rightarrow(Y,v)$ is fuzzy continuous, then $f^{-1}(F)$ is a fuzzy closed subset of (X, u) for every fuzzy closed subset F of (Y, v).

Definition 2.5 [6]

Let (X, u) and (Y, v) be fuzzy closure spaces. A map $f:(X,u)\rightarrow(Y,v)$ is said to be fuzzy closed (resp.fuzzy open) if $f(F)$ is a fuzzy closed (resp.fuzzy open) subset of (Y, v) whenever F is a fuzzy closed (resp. fuzzy open) subset of (X, u).

Definition 2.6 [8]

A *fuzzy biclosure space* is a triple (X,u_1,u_2) where X be a set and u_1,u_2 are two fuzzy closure operators on X.

Definition 2.7 [8]

A subset A of a fuzzy biclosure space (X,u_1,u_2) is called fuzzy closed if $u_1u_2A=A$. The complement of fuzzy closed set is called fuzzy open.

Clearly, A is a fuzzy closed subset of fuzzy biclosure space (X,u_1,u_2) if and only if A is both a fuzzy closed subset of (X,u_1) and (X,u_2).

Let A be a fuzzy closed subset of a fuzzy biclosure space(X,u_1,u_2). The following conditions are equivalent

(i) $u_2u_1A=A$

(ii) $u_1A=A$, $u_2A=A$.

The following statement is obivious :

Proposition 2.8 [8]

Let (X,u_1,u_2) be a fuzzy biclosure space and let $A\leq X$ .Then

(i) A is fuzzy open if and only if $A=X-u_1u_2(X-A)$.

(ii) If G is fuzzy open and $G\leq A$, then $G\leq X-u_1u_2(X-A)$.

Definition 2.9 [8]

Let (X,u_1,u_2) be fuzzy biclosure space. A fuzzy biclosure space (Y,v_1,v_2) is called a fuzzy subspace of (X,u_1,u_2) if $Y \leq X$ and $v_i A = u_i A \wedge Y$ for each $i \in \{1,2\}$ and each subset $A \leq Y$.

Proposition 2.10 [8]

Let (X,u_1,u_2) be fuzzy biclosure space and let (Y,v_1,v_2) be a fuzzy closed subspace of (X,u_1,u_2). If F is a fuzzy closed subset of (Y,v_1,v_2), then F is a fuzzy closed subset of (X,u_1,u_2).

Proposition 2.11 [8]

Let $\{(X_\alpha,u_\alpha{}^1,u_\alpha{}^2):\alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then F is a fuzzy closed subset of $(X_\beta,u_\beta{}^1,u_\beta{}^2)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy closed subset of $\prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} (X_\alpha,u_\alpha{}^1,u_\alpha{}^2)$.

Proposition 2.12 [8]

Let $\{(X_\alpha,u_\alpha{}^1,u_\alpha{}^2):\alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then G is a fuzzy open subset of $(X_\beta,u_\beta{}^1,u_\beta{}^2)$ if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy open subset of $\prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} (X_\alpha,u_\alpha{}^1,u_\alpha{}^2)$.

3. Normal Fuzzy Biclosure Spaces

In this section we introduce the concept of normal fuzzy biclosure spaces and study some of their properties.

Definition 3.1

A biclosure space (X,u_1,u_2) is said to be normal fuzzy biclosure space if, for every disjoint fuzzy closed subset H of (X,u_1) and fuzzy closed subset K of (X,u_2) there exist disjoint fuzzy open subset U of and (X,u_1) fuzzy open subset of such that $H \leq U$ and $K \leq V$.

Example 3.2.

Let $X=\{a,b\}$. For any $A \in I^X$, let supp $A=\{x \in X : A(x)>0\}$. Define fuzzy closure operators $u_1, u_2 : I^X \to I^X$ by the following (for simplicity, we identify each ordinary subset of X with its characteristic function) :

$u_1 A = u_2 A = \{a\}$ if supp $A = \{a\}$

$u_1 A = u_2 A = \{b\}$ if supp $A = \{b\}$

$u_1 A = u_2 A = X$ if supp $A = X$

$u_1 A = u_2 A = \phi$ if supp $A = \phi$

Then (X, u_1, u_2) is a normal fuzzy biclosure space.

Lemma 3.3

Let (X, u_1, u_2) be a fuzzy biclosure space and let (Y, v_1, v_2) be a fuzzy closed subspace of (X, u_1, u_2) If G is a fuzzy open subset of (X, u_1) and a fuzzy open subset of (X, u_2), then $G \wedge Y$ is a fuzzy open subset of (Y, v_1) and a fuzzy open subset of (Y, v_2).

Proof

Let G be a fuzzy open subset of (X, u_1).

Since $Y-(G \wedge Y) = Y \wedge (X-G) = u_1 Y \wedge u_1(X-G) = u_1(Y \wedge (X-G))$ $= u_1(Y \wedge (X-G)) \wedge Y = v_1(Y \wedge (X-G)) = v_1(Y-(G \wedge Y))$. Hence, $(Y-(G \wedge Y))$ is a fuzzy closed subset of (Y, v_1). Consequently $G \wedge Y$ is a fuzzy open subset of (Y, v_1). Similarly, if G is a fuzzy open subset of (X, u_2), then $G \wedge Y$ is a fuzzy open subset of (Y, v_2).

Lemma 3.4

Let (X, u_1, u_2) be a fuzzy biclosure space and let (Y, v_1, v_2) be a fuzzy closed subspace of (X, u_1, u_2). If F is a fuzzy closed subset of (Y, v_1, v_2), then F is a fuzzy closed subset of (X, u_1, u_2).

Proof

Let F be a fuzzy closed subset of (Y, v_1, v_2). Then F is both a fuzzy closed subset of of (Y, v_1) and (Y, v_2). Consequently $F = v_1 F = u_1 F \wedge Y = u_1 F \wedge u_1 Y = u_1(F \wedge Y) = u_1 F$ and $F = u_2 F$. Therefore, F is both a fuzzy closed subset of (X, u_1) and (X, u_2). Hence F is a fuzzy closed subset of (X, u_1, u_2).

Proposition 3.5

Let (X, u_1, u_2) be a fuzzy biclosure space and let (Y, v_1, v_2) be a fuzzy closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a normal fuzzy biclosure space, then (Y, v_1, v_2) is a normal fuzzy biclosure space.

Proof

Let A be a fuzzy closed subset of (Y,v_1) and B be a fuzzy closed subset of (Y,v_2) such that $A\wedge B=\phi$.By Lemma 3.4 , A is a fuzzy closed subset of (X,u_1) and B is a fuzzy closed subset of (X,u_2). Since (X,u_1,u_2) is a normal fuzzy biclosure space , there exist fuzzy open subsets U of (X,u_1) and a fuzzy open subsets V of (X,u_2) such that $A\leq U$, $B\leq V$ and $U\wedge V=\phi$.Consequently, $A\leq U\wedge Y$, and $(U\wedge Y)\wedge(V\wedge Y)=\phi$. By Lemma 3.3 , $U\wedge Y$ is a fuzzy open subsets of (Y,v_1) and $V\wedge Y$ is a fuzzy open subsets of (Y,v_2) .Hence (Y,v_1,v_2) is a normal fuzzy biclosure space.

Proposition 3.6

Let $\{(X_\alpha,u_\alpha^1,u_\alpha^2):\alpha\in J\}$ be a family of fuzzy biclosure spaces. Then $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$ is normal fuzzy biclosure space if and only if $(X_\alpha,u_\alpha^1,u_\alpha^2)$ is a normal fuzzy biclosure space for each $\alpha\in J$.

Proof

Suppose that $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$ is normal fuzzy biclosure space. Let $\beta\in J$ and let F be fuzzy closed subset of (X,u_β^1) and F' be fuzzy closed subset of (X,u_β^2) such that $F\wedge F'=\phi$. Then $F\times\prod_{\alpha\in J}X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1)$ and $F'\times\prod_{\alpha\in J}X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^2)$ such that $(F\times\prod_{\alpha\in J}X_\alpha)\wedge(F'\times\prod_{\alpha\in J}X_\alpha)=\phi$. Since $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1,u_\alpha^2)$ is a normal fuzzy biclosure space , there exist a fuzzy open subset U of (X_β,u_β^1) and a fuzzy open subset V of (X_β,u_β^2) such that $F\leq U$, $F'\leq V$ and $U\wedge V=\phi$. Hence , $(X_\beta,u_\beta^1,u_\beta^2)$ is a normal fuzzy biclosure space .

Conversely, Suppose that $(X_\alpha,u_\alpha^1,u_\alpha^2)$ is a normal fuzzy biclosure space for each $\alpha\in J$. Let F be a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^1)$ and F' be a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,u_\alpha^2)$ such that $F\wedge F'=\phi$.Then $\pi_\beta(F)$ is a fuzzy closed subset of (X_β,u_β^1) and $\pi_\beta(F')$ is a fuzzy closed subset of

(X_β, u_β^2). Since $(X_\beta, u_\beta^1, u_\beta^2)$ is a normal fuzzy biclosure space, there exist fuzzy open subset U of of (X_β, u_β^1) and fuzzy open subset V of (X_β, u_β^2) such that $F \le U$, $F' \le V$ and $U \wedge V = \phi$. Therefore, $F \le \pi^{-1}{}_\beta(U) = U \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ and $F' \le \pi_\beta{}^{-1}(V) = V \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$. Consequently, $U \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is fuzzy open subset of (X_α, u_α^1) and $V \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy open subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^2)$ such that $(U \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha) \wedge (V \times \prod_{\substack{\alpha \ne \beta \\ \alpha \in J}} X_\alpha) = \phi$. Hence, $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a normal fuzzy biclosure space.

Proposition 3.7

Let (X, u_1, u_2) and (Y, v_1, v_2) be fuzzy biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be injective, fuzzy closed and fuzzy continuous. If (Y, v_1, v_2) is normal fuzzy biclosure space, then (X, u_1, u_2) is a normal fuzzy biclosure space.

Proof

Let F be fuzzy closed subset of (X, u_1) and F' be fuzzy closed subset of (X, u_2) such that $F \wedge F' = \phi$. Since f is injective and fuzzy closed, $f(F)$ is a fuzzy closed subset of (Y, v_1) and is a fuzzy closed subset of (Y, v_2) such that $f(F) \wedge f(F') = \phi$. Since (Y, v_1, v_2) normal fuzzy biclosure space, there exists a disjoint fuzzy open subset U of and (Y, v_1) and a fuzzy open subset V of (Y, v_2) such that $f(F) \le U$ and $f(F') \le V$. Since f is fuzzy continuous, $f^{-1}(U)$ is a fuzzy open subset of (X, u_1) and $f^{-1}(V)$ is a fuzzy open subset of (X, u_2) such that $F \le f^{-1}(U)$, $F' \le f^{-1}(V)$ and $f^{-1}(U) \wedge f^{-1}(V) = \phi$. Hence, (X, u_1, u_2) is a normal fuzzy biclosure space.

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