Fuzzy Strong n-Inner Product Space

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Abstract

The purpose of this paper is to introduce the notion of fuzzy strong n-inner product space as a generalization of fuzzy n-inner product space. Ascending family of strong α -n-inner products corresponding to fuzzy strong n-inner product is introduced and we provide some results on it.

Key words and phrases: n-inner product, fuzzy n-inner product, fuzzy strong n-inner product, α -n inner product.

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1. Introduction

Significant contribution in the theory of 2-inner product space and n-inner product space has been made by eminent researchers in [3, 4, 5, 6, and 7]. Recently, Vijayabalaji and Thillaigovindan have introduced the notion of fuzzy n-inner product space in [9]. In this paper as a natural generalization of fuzzy n-inner product space we introduce the notion of fuzzy strong n- inner product space. Vijayabalaji and Thillaigovindan raised a problem of constructing α-n-inner product space Studied in [9] and answer to this problem is provided in this paper by constructing α - strong n-inner product space.

Analogue of α-n-inner product space in [9] we introduce the notion of α-strong n-inner product space. We then interrelate α-strong n- inner product space and α-n-normed linear space.

2. Preliminaries

In this section we recall some concepts which will be needed in the sequel.

Definition 2.1 [2]

Let n be a natural number greater than 1 and X be a real linear space of dimension greater than or equal to n and let $(\bullet,\bullet|\bullet,...,\bullet)$ be a real valued function on $\underbrace{X\times X...\times X}_{(n+1)times}=X^{n+1}$ satisfying the following conditions :

(1) (i) $(x,x|x_2,...,x_n)\geq 0$,

(ii) $(x,x|x_2,...,x_n)=0$ if any only if $x,x_2,...,x_n$ are linearly dependent,

(2) $(x,y|x_2,...,x_n)=(y,x|x_2,...,x_n)$,

(3) $(x,y|x_2,...,x_n)$ is invariant under any permutation of $x,x_2,...,x_n$,

(4) $(x,x|x_2,...,x_n)=(x_2,x_2|x,x_3,...,x_n)$,

(5) $(ax,x|x_2,...,x_n)=a(x,x|x_2,...,x_n)$,

(6) $(x+x',y|x_2,...,x_n)=(x,y|x_2,...,x_n)+(x',y|x_2,...,x_n)$

Then $(\bullet,\bullet|\bullet,...,\bullet)$ is called an n-inner product on X and $(X,(\bullet,\bullet|\bullet,...,\bullet))$ is called an n-inner product space.

Definition 2.2 [8]

Let n be a natural number greater than 1 and X be a real linear space of dimension greater than or equal to n . A real valued function $\|\bullet,\bullet,...,\bullet\|$ on $\underbrace{X\times X...\times X}_{ntimes}=X^n$ satisfying the following conditions

(1) $\|x_1,x_2,...,x_n\|=0$ if any only if $x_1,x_2,...,x_n$ are linearly dependent,

(2) $\|x_1,x_2,...,x_n\|$ is invariant under any permutation,

(3) $\|x_1,x_2,...,ax_n\|=|a|\ \|x_1,x_2,...,x_n\|$, for any $\alpha\in R$ (real),

(4) $\|x_1,x_2,...,x_{n-1},y+z\leq\ \|x_1,x_2,...,x_{n-1},y\|+\|x_1,x_2,...,x_{n-1},z\|$,

is called an n-norm on X and the pair $(X,\|\bullet,\bullet,...,\bullet\|)$ is called an n-normed linear Space.

Remark 2.3 [8]

In the above definition if we replace (3) by,

(3)' $\| x_1, x_2, ..., ax_n \| = |a|^p \ \| x_1, x_2, ..., x_n \|$, for any $\alpha \in R$ (real) and $0 \le p \le 1$,

then (X, $\| \bullet, \bullet, ..., \bullet \|$) is called as quasi n-normed linear space.

Remark 2.4 [3]

If an n-inner product space (X, $(\bullet, \bullet | \bullet, ..., \bullet)$) is given then $\| x_1, x_2, ..., x_n \| = \sqrt{(x_1, x_1 | x_2, ..., x_n)}$ defines an n-norm on X. Further the following extension of Cauchy-Buniakowski inequality is also true $|(x, y | x_2, ..., x_n)| \le \sqrt{(x, x | x_2, ..., x_n)} \ \sqrt{(y, y | x_2, ..., x_n)}$.

Definition 2.5 [8]

Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times R$ (R -set of real numbers) is called a fuzzy n-norm on X if and only if :

(N1) For all $t \in R$ with $t \le 0$, $N(x_1, x_2, ..., x_n, t) = 0$.

(N2) For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(N3) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(N4) For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \ne 0, c \in F$ (field).

(N5) For all $s, t \in R$, $N(x_1, x_2, ..., x_n + x_n', s + t) \ge \min\{N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x_n', t)\}$.

(N6) $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 2.6 [8]

In the above definition if we replace (N4) by,

(N4)' For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|^p})$, if $c \ne 0, c \in F$ (field), $0 \le p \le 1$. Then (X, N) is called a fuzzy quasi n-normed linear space or in short f-q-n-NLS.

Theorem 2.7 [8]

Let (X, N) be a f-n-NLS. Assume the condition that

(N^*) $N(x_1, x_2, ..., x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent. Define

$\| x_1, x_2, ..., x_n \|_\alpha = \inf \{ t : N(x_1, x_2, ..., x_n, t) \geq \alpha \}$, $\alpha \in (0,1)$. Then $\{ \| \bullet, \bullet, ..., \bullet \|_\alpha : \alpha \in (0,1) \}$ is an ascending family of n-norms on X. We call these n-norms as α -n-norms on X corresponding to the fuzzy n-norm on X.

Definition 2.8 [9]

Let X be a linear space over a real field F. A fuzzy subset

$J : X^{n+1} \times R \to [0,1]$ (R -set of real numbers) is called a fuzzy n-m on X if and only if:

(1) For all $t \in R$ with $t \leq 0$, $J(x, x \mid x_2, ..., x_n, t) = 0$.

(2) For all $t \in R$ with $t > 0$, $J(x, x \mid x_2, ..., x_n, t) = 1$ if and only if $x, x_2, ..., x_n$ are linearly dependent.

(3) For all $t > 0$, $J(x, y \mid x_2, ..., x_n, t) = J(y, x \mid x_2, ..., x_n, t)$.

(4) $J(x, y \mid x_2, ..., x_n, t)$ is invariant under any permutation of $x_2, ..., x_n$.

(5) For all $t > 0$, $J(x, x \mid x_2, ..., x_n, t) = J(x_2, x_2 \mid x, ..., x_n, t)$

(6) For all $t > 0$, $J(ax, bx \mid x_2, ..., x_n, t) = J(x, x \mid x_2, ..., x_n, \frac{t}{|ab|}), a, b \in R$ (real).

(7) For all $s, t \in R$, $J(x + x', y \mid x_2, ..., x_n, t + s) \geq \min \{ J(x, y \mid x_2, ..., x_n, t), J(x', y \mid x_2, ..., x_n, s) \}$.

(8) For all $s, t \in R$ with $s > 0, t > 0$, $J(x, y \mid x_2, ..., x_n, \sqrt{ts}) \geq$
$\min \{ J(x, x \mid x_2, ..., x_n, t), J(y, y \mid x_2, ..., x_n, s) \}$,

(9) $J(x, y \mid x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} J(x, y \mid x_2, ..., x_n, t) = 1$.

Then (X, J) is called a fuzzy n-inner product space or in short f-n-IPS.

3. Fuzzy strong n-inner product space

By generalizing Definition 2.8 we obtain the new notion of fuzzy strong n-inner product space as follows.

Definition 3.1

Let X be a linear space over a real field F. A fuzzy subset $J : X^{n+1} \times R \to [0,1]$ (R -set of real numbers) is called a fuzzy strong n-inner product on X if and only if:

(1) For all $t \in R$ with $t \leq 0$, $J(x, x \mid x_2, ..., x_n, t) = 0$.

(2) For all $t \in R$ with $t > 0$, $J(x, x \mid x_2, ..., x_n, t) = 1$ if and only if $x, x_2, ..., x_n$ are linearly dependent.

(3) For all $t > 0$, $J(x, y \mid x_2, ..., x_n, t) = J(y, x \mid x_2, ..., x_n, t)$.

(4) $J(x,y \mid x_2,\ldots,x_n,t)$ is invariant under any permutation of $x_2,\ldots,x_n$.

(5) For all $t>0$, $J(x,x \mid x_2,\ldots,x_n,t)=J(x_2,x_2 \mid x,\ldots,x_n,t)$

(6) For all $t>0$, $J(ax,bx \mid x_2,\ldots,x_n,t)= J(x,x \mid x_2,\ldots,x_n,\frac{t}{|ab|}), a,b \in R$ (real).

(7) For all $s,t \in R$, $J(x+x',y \mid x_2,\ldots,x_n,t+s)=$ min $\{ J(x,y \mid x_2,\ldots,x_n,t), J(x',y \mid x_2,\ldots,x_n,s) \}$.

(8) For all $s,t \in R$ with $s>0, t>0$, $J(x,y \mid x_2,\ldots,x_n,\sqrt{ts})=$

$\min\{ J(x,x \mid x_2,\ldots,x_n,t), J(y,y \mid x_2,\ldots,x_n,s) \}$,

(9) $J(x,y \mid x_2,\ldots,x_n,t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} J(x,y \mid x_2,\ldots,x_n,t)=1$.

Then (X,J) is called a fuzzy strong n-inner product space or in short f-ST-n-IPS.

Example 3.2

Let $(X,(\bullet,\bullet \mid \bullet,\ldots,\bullet))$ be an n-inner product space. Define

$$J(x,y \mid x_2,\ldots,x_n,t)=\begin{cases}\dfrac{t}{t+|(x,y \mid x_2,\ldots,x_n)|}, & \text{when } t>0, t\in R, (x,y \mid x_2,\ldots,x_n)\in X^{n+1}\\ o, otherwise. & \end{cases}$$

Then (X,J) is a f-ST-n-IPS.

Proof

As given in [9].

4. Strong α - n-inner product space

Analogue of ascending family of quasi -n-norms corresponding to the fuzzy quasi n-norm that was introduced in [9], we now introduce the new notion of strong -n-inner product corresponding to fuzzy strong n-inner product in the following theorem.

Theorem 4.1

Let (X,J) be a f-ST-n-IPS. Assume the condition that

(10) $J(x,y \mid x_2,\ldots,x_n,t)>0$ implies $x,x_2,\ldots,x_n$ are linearly dependent.

Define $(x,x \mid x_2,\ldots,x_n)_\alpha = \inf\{ t : J(x,x \mid x_2,\ldots,x_n,t)\geq \alpha \}\ \alpha\in(0,1)$.

Then $\{ (\bullet,\bullet \mid \bullet,\ldots,\bullet)_\alpha : \alpha \in (0,1) \}$, is an ascending family of strong n-inner products on X.

We call these n-inner products as strong α -n-inner product on X corresponding to the fuzzy strong n-inner product on X.

Proof

(1) (i) Clearly $(x,x \mid x_2,\ldots,x_n)_\alpha \geq 0$,

(ii) $(x,x \mid x_2,\ldots,x_n)_\alpha = 0$

$\Rightarrow \inf\{ t : J(x,x \mid x_2,\ldots,x_n,t) \geq \alpha \} = 0$

$\Rightarrow$ For all $t \in R, t > 0$, $J(x,x \mid x_2,\ldots,x_n,t) \geq \alpha > 0$, $\alpha \in (0,1)$.

$\Rightarrow$ By (10) $x, x_2,\ldots,x_n$ are linearly dependent.

Conversely assume that $x, x_2,\ldots,x_n$ are linearly dependent.

$\Rightarrow$ By (2) $J(x,x \mid x_2,\ldots,x_n,t) = 1$ for all $t > 0$

$\Rightarrow$ For all $\alpha \in (0,1)$ $\inf\{ t : J(x,x \mid x_2,\ldots,x_n,t) \geq \alpha \} = 0$

$\Rightarrow$ $(x,x \mid x_2,\ldots,x_n)_\alpha = 0$.

(2) As $J(x,y \mid x_2,\ldots,x_n,t)$ is invariant under any permutation of $x, x_2,\ldots,x_n$ it follows that $(x,x \mid x_2,\ldots,x_n)_\alpha$ is invariant under any permutation.

(3) For all $c \in F$,

$(cx,x \mid x_2,\ldots,x_n)_\alpha$

$= \inf \{ s : J(cx,x \mid x_2,\ldots,x_n,t) \geq \alpha \}$

$= \inf \{ s : J(x,x \mid x_2,\ldots,x_n,\frac{s}{|c|}) \geq \alpha \}$.

Let $t = \frac{s}{|c|}$ then,

$(cx,x \mid x_2,\ldots,x_n)_\alpha$

$= \inf \{ |c|t : J(x,x \mid x_2,\ldots,x_n,t) \geq \alpha \}$

$= |c| \inf \{ t : J(x,x \mid x_2,\ldots,x_n,t) \geq \alpha \}$

$= |c| (x,x \mid x_2,\ldots,x_n)_\alpha$.

(4) $(x,y \mid x_2,\ldots,x_n)_\alpha + (x',y \mid x_2,\ldots,x_n)_\alpha$

$= \inf \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha \} + \inf \{ t : J(x',y \mid x_2,\ldots,x_n,t) \geq \alpha \}$

$$= \inf \{ t+s : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha, J(x',x \mid x_2,\ldots,x_n,s) \geq \alpha \}$$

$$= \inf \{ t+s : J(x+x',y \mid x_2,\ldots,x_n,t+s) \geq \alpha \}$$

$$= \inf \{ r : J(x+x',y \mid x_2,\ldots,x_n,r) \geq \alpha \}, r = t+s$$

$$= (x+x',y \mid x_2,\ldots,x_n)_\alpha .$$

Therefore $(x+x',y \mid x_2,\ldots,x_n)_\alpha = (x,y \mid x_2,\ldots,x_n)_\alpha + (x',y \mid x_2,\ldots,x_n)_\alpha$.

Thus $\{ (\bullet,\bullet \mid \bullet,\ldots,\bullet)_\alpha : \alpha \in (0,1) \}$, is a strong n-inner product on X.

Let $0 < \alpha_1 < \alpha_2$. Then

$$(x,y \mid x_2,\ldots,x_n)_{\alpha 1} = \inf \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha_1 \}$$

$$(x,y \mid x_2,\ldots,x_n)_{\alpha 2} = \inf \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha_2 \}.$$

As $\alpha_1 < \alpha_2$

$$\Rightarrow \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha_2 \} \subset \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha_1 \}$$

$$\Rightarrow \inf \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha_2 \} \geq \inf \{ t : J(x,y \mid x_2,\ldots,x_n,t) \geq \alpha_1 \}$$

$$\Rightarrow (x,y \mid x_2,\ldots,x_n)_{\alpha 2} \geq (x,y \mid x_2,\ldots,x_n)_{\alpha 1}.$$

Hence $\{ (\bullet,\bullet \mid \bullet,\ldots,\bullet)_\alpha : \alpha \in (0,1) \}$, is an ascending family of strong n-inner products corresponding to the fuzzy strong n-inner product on X.

Inspired by the theory of α -n-normed linear space that was introduced in [8], we now inter-relate -n-norm and strong α-n-inner product in the following theorem.

Further we prove the Cauchy-Buniakowski inequality for strong α-n-inner product space in the following theorem.

Theorem 4.2

Let $\{X,(\bullet,\bullet \mid \bullet,\ldots,\bullet)\}$, be a strong α -n-inner product space. Then the following statements are equivalent.

(1) $\| x,x_2,\ldots,x_n \|_\alpha = (x,x \mid x_2,\ldots,x_n)_\alpha^{\frac{1}{2}}$ defines an α -n-norm on X.

(2) $|(x,y \mid x_2,\ldots,x_n)_\alpha| \leq (x,x \mid x_2,\ldots,x_n)_\alpha^{\frac{1}{2}} \; (y,y \mid x_2,\ldots,x_n)_\alpha^{\frac{1}{2}}$

Proof

(1) $\Rightarrow$ (2)

Assume (1) is true. We have to prove (2).

If $y=0$, then equality holds in (2). Consider,

$$0 \le \| \, |x-\lambda y, x_2,\ldots,x_n \, \|_\alpha^{\,2}$$

$$=(x-\lambda y, x-\lambda y \mid x_2,\ldots,x_n)_\alpha$$

$$=(x,x \mid x_2,\ldots,x_n)_\alpha - \bar{\lambda}(x,y \mid x_2,\ldots,x_n)_\alpha - \lambda(y,x \mid x_2,\ldots,x_n)_\alpha + \lambda\bar{\lambda}(y,y \mid x_2,\ldots,x_n)_\alpha .$$

Let $\lambda = \dfrac{(x,y \mid x_2,\ldots,x_n)_\alpha}{(y,y \mid x_2,\ldots,x_n)_\alpha}$

$$\Rightarrow 0 \le \| \, x,x_2,\ldots,x_n \, \|_\alpha^{\,2} - \frac{|(x,y \mid x_2,\ldots,x_n)_\alpha|^2}{\| \, y,x_2,\ldots,x_n \, \|_\alpha^{\,2}} - \frac{|(x,y \mid x_2,\ldots,x_n)_\alpha|^2}{\| \, y,x_2,\ldots,x_n \, \|_\alpha^{\,2}} - \frac{|(x,y \mid x_2,\ldots,x_n)_\alpha|^2}{\| \, y,x_2,\ldots,x_n \, \|_\alpha^{\,2}}$$

$$- \frac{|(x,y \mid x_2,\ldots,x_n)_\alpha|^2}{\| \, y,x_2,\ldots,x_n \, \|_\alpha^{\,4}} \| \, y,x_2,\ldots,x_n \, \|_\alpha^{\,2}$$

$$\Rightarrow 0 \le \| \, x,x_2,\ldots,x_n \, \|_\alpha^{\,2} - \frac{|(x,y \mid x_2,\ldots,x_n)_\alpha|^2}{\| \, y,x_2,\ldots,x_n \, \|_\alpha^{\,2}}$$

$$\Rightarrow \| \, x,x_2,\ldots,x_n \, \|_\alpha^{\,2} \| \, y,y_2,\ldots,y_n \, \|_\alpha^{\,2} \ge | \, (x,y \mid x_2,\ldots,x_n)_\alpha \, |^2$$

$$\Rightarrow |(x,y \mid x_2,\ldots,x_n)_\alpha| \le (x,x \mid x_2,\ldots,x_n)_\alpha^{\frac{1}{2}} .$$

(2) $\Rightarrow$ (1)

(i) Clearly

$$\| \, x_1,x_2,\ldots,x_n \, \|_\alpha = 0$$

$$\Leftrightarrow (x_1,x_1 \mid x_2,\ldots,x_n)_\alpha^{\frac{1}{2}} = 0$$

$\Leftrightarrow x_1,x_2,\ldots,x_n$ are linearly dependent.

(ii) As $(x_1,x_1 \mid x_2,\ldots,x_n)_\alpha$ is invariant under any permutation of $x_1,x_2,\ldots,x_n$ it follows

$\| \, x_1,x_2,\ldots,x_n \, \|_\alpha$ is invariant under any permutation of $x_1,x_2,\ldots,x_n$.

(iii) $\| \, cx_1,x_2,\ldots,x_n \, \|_\alpha$

$$=(cx_1,cx_1 \mid x_2,\ldots,x_n)_\alpha$$

$$= c\bar{c}(x_1, x_1 \mid x_2, \ldots, x_n)_\alpha$$

$$= |\ c\ |^2\ (x_1, x_1 \mid x_2, \ldots, x_n)_\alpha .$$

Taking square root,

$$\Rightarrow \| cx_1, x_2, \ldots, x_n \|_\alpha = |\ c\ | \| x_1, x_2, \ldots, x_n \|_\alpha .$$

(iv) $\| x + y, x_2, \ldots, x_n \|_\alpha^{\ 2}$

$$= (x + y, x + y \mid x_2, \ldots, x_n)_\alpha$$

$$= (x, x \mid x_2, \ldots, x_n)_\alpha + (x, y \mid x_2, \ldots, x_n)_\alpha + (y, x \mid x_2, \ldots, x_n)_\alpha + (y, y \mid x_2, \ldots, x_n)_\alpha$$

$$= (x, x \mid x_2, \ldots, x_n)_\alpha + (x, y \mid x_2, \ldots, x_n)_\alpha + 2\ \mathrm{re}\,(x, y \mid x_2, \ldots, x_n)_\alpha$$

$$\leq \| x, x_2, \ldots, x_n \|_\alpha^{\ 2} + \| y, x_2, \ldots, x_n \|_\alpha^{\ 2} + 2 \| x, x_2, \ldots, x_n \|_\alpha \| y, x_2, \ldots, x_n \|_\alpha$$

$$\leq \{ \| x, x_2, \ldots, x_n \|_\alpha + \| y, x_2, \ldots, x_n \|_\alpha \}^2$$

$$\Rightarrow \| x + y, x_2, \ldots, x_n \|_\alpha \leq \| x, x_2, \ldots, x_n \|_\alpha + \| y, x_2, \ldots, x_n \|_\alpha .$$

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