Reduction of Linear Dynamic Systems using Dominant Pole Retention Method and Modified Cauer Continued Fraction

G. Parmar[1], Dr. R. Prasad[2], Dr. S. Mukherjee [3]

ABSTRACT

The authors present an algorithm for obtaining stable reduced order models using the combined advantages of the dominant pole retention method and the modified Cauer continued fraction. The reduction procedure is simple and computer oriented. It is shown that the method has several advantages, e.g. the reduced order models retain the steady-state value and stability of the original system. The proposed method has also been extended for the order reduction of linear multivariable systems. Three numerical examples are solved to illustrate the superiority of the method over some existing ones including one example of multivariable system.

Keywords: Cauer continued fraction, Dominant pole, Integral square error, Multivariable system, Order reduction, Stability.

1. INTRODUCTION

The exact analysis of complex systems is difficult and possibly not desirable on economic and computational considerations. This makes apparent the need for using reduced order models which constitute a good approximation of the original system. Numerous methods are available in the literature for order-reduction of linear continuous systems in time domain as well as in frequency domain [1-6]. Some trivial extensions of single input single output (SISO) methods to reduce multiinput multioutput (MIMO) systems have also been carried out in [7-10]. Each of these methods has both advantages and disadvantages when tried on a particular system. In spite of several methods available, no approach always gives the best results for all systems.

The problem of overcoming the instability of reduced order models derived through continued fraction technique has been investigated in [11-13]. In [11], the denominator of the reduced order model is formed by Routh array, while in [12, 13] the stability equation method is used for the same purpose. Then the numerator dynamics is chosen to fit a given number of continued fraction quotients.

In this paper, the authors present a new algorithm for order reduction, which combines the advantages of the dominant pole retention method and modified Cauer continued fraction technique. The proposed method consists of retaining the dominant poles of the original system, where zeros are synthesized by using the modified Cauer continued fraction method (MCF). The method has also been extended for order reduction of linear multivariable systems. In the following sections, the method is described in detail with the help of three numerical examples.

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2. Description Of The Method

Let, the n[th] high order system (HOS) $G_n(s)$ and its r[th] low order system (LOS) be represented by :

$$G_n(s) = \frac{b_{11} + b_{12}s + b_{13}s^2 + ... + b_{1,n}s^{n-1}}{a_{11} + a_{12}s + a_{13}s^2 + ... + a_{1,n}s^{n-1} + s^n} \quad (1)$$

$$G_r(s) = \frac{q_{11} + q_{12}s + ... + q_{1,r}s^{r-1}}{(s+\lambda_1)\ (s+\lambda_2)\ ......(s+\lambda_r)} \quad (2)$$

where, $\lambda_1, \lambda_2, ..., \lambda_r$ are the dominant poles of the HOS.

or,

$$G_r(s) = \frac{q_{11} + q_{12}s + ... + q_{1,r}s^{r-1}}{p_{11} + p_{12}s + ... + p_{1,r}s^{r-1} + s^r} \quad (3)$$

Further, the method consists of following steps :

Step-1:

Retention of dominant poles of HOS in LOS [14, 15] : Depending on the order to be reduced to, the poles nearest to the origin are retained. This implies that the over all behavior of the reduced system will be very similar to the original system, since the contribution of the unretained eigen values to the system response are important only at the beginning of the response, where as the eigen values retained are important throughout the whole of the response, and, infact, determine the type of the response of the system.

Therefore, the denominator polynomial in (2) is now known, which is given by

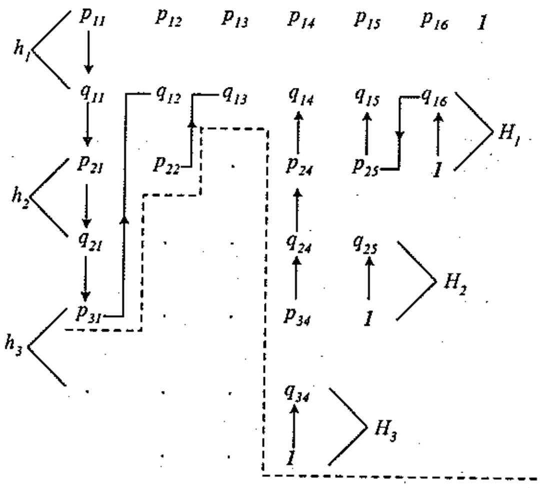
$$D_r(s) = p_{11} + p_{12}s + ... + p_{1,r}s^{r-1} + s^r \quad (4)$$

Step-2:

By applying the algorithm given in [16], the first 'r' quotients of modified Cauer form of continued fraction, viz. h_1, H_1, h_2, H_2,....are evaluated.

Step-3:

Now a modified Routh array for r = 6 is built as given below :



where, the first two rows are formed from the denominator and numerator coefficients of $G_r(s)$ in (3) and the remaining entries in the array are obtained by the algorithm given in [11]. The sequence of computation is indicated by the arrows.

3. Extension To Multivariable Systems

Let, the transfer matrix of the HOS of order 'n' having 'p' inputs and 'm' outputs is :

$$[G(s)] = \frac{1}{D_n(s)} \begin{bmatrix} a_{11}(s) & a_{12}(s) & a_{13}(s) & ... & a_{1p}(s) \\ a_{21}(s) & a_{22}(s) & a_{23}(s) & ... & a_{2p}(s) \\ \vdots & \vdots & \vdots & ... & \vdots \\ a_{m1}(s) & a_{m2}(s) & a_{m3}(s) & ... & a_{mp}(s) \end{bmatrix}$$

or, $[G(s)] = [g_{ij}(s)]$, $i = 1, 2, ..., m;\ j = 1, 2, ..., p$ is a $m \times p$ transfer matrix.

The general form of $g_{ij}(s)$ of $[G(s)]$ in (6) is taken as

$$g_{ij}(s) = \frac{a_{ij}(s)}{D_n(s)}$$

$$= \frac{b_{11} + b_{12}s + b_{13}s^2 + ... + b_{1,n}s^{n-1}}{a_{11} + a_{12}s + a_{13}s^2 + ... + a_{1,n}s^{n-1} + s^n} \text{ or,}$$

$$g_{ij}(s) = \frac{b_{11} + b_{12}s + b_{13}s^2 + ... + b_{1,n}s^{n-1}}{(s+\lambda_1)\ (s+\lambda_2)\ ......(s+\lambda_n)} \text{ where}$$

$-\lambda_1 < -\lambda_2 < ...... < -\lambda_n$ are poles of the HOS.

Let, the transfer matrix of the LOS of order 'r' having 'p' inputs and 'm' outputs to be synthesized is :

$$[R(s)] = \frac{1}{D_r(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) & b_{13}(s) & \dots & b_{1p}(s) \\ b_{21}(s) & b_{22}(s) & b_{23}(s) & \dots & b_{2p}(s) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{m1}(s) & b_{m2}(s) & b_{m3}(s) & \dots & b_{mp}(s) \end{bmatrix}$$

or $[R(s)] = [r_{ij}(s)],\ i = 1, 2, \dots\dots, m;\ j = 1, 2, \dots\dots, p$

is a $m \times p$ transfer matrix.

The general form of $r_{ij}(s)$ of $[R(s)]$ in (9) is taken as

$$r_{ij}(s) = \frac{b_{ij}(s)}{D_r(s)}$$

$$= \frac{q_{11} + q_{12}s + \dots\dots + q_{1,r}s^{r-1}}{(s+\lambda_1)\ (s+\lambda_2)\ \dots\dots(s+\lambda_r)} \qquad (10)$$

where, $-\lambda_1 < -\lambda_2 < \dots\dots < -\lambda_r$ are the dominant poles of the HOS.

The proposed method consists of retaining the dominant poles of the original system, where zeros are synthesized by using the modified Cauer continued fraction method (MCF).

Basically; the method starts with fixation of the denominator of the LOS by dominant pole retention method followed by the determination of coefficients of the numerator polynomials of each element of the LOS transfer matrix by matching the quotients of modified Cauer continued fraction method.

4. Numerical Examples

Three numerical examples are chosen from the literature for the comparison of the reduced order models (LOS) with the original system (HOS). The proposed method is described in detail for one example while only the results of the other examples are given.

An error index I.S.E. [5] known as integral square error in between the transient parts of original and reduced order systems is calculated to measure the goodness of the LOS (i.e. the smaller the ISE, the closer is $G_r(s)$ to $G_n(s)$), which is given by :

$$\text{I.S.E.} = \int_0^\infty [y(t) - y_r(t)]^2\, dt \qquad (11)$$

where, $y(t)$ and $y_r(t)$ are the unit step responses of original and reduced order systems.

Example-1. Consider a 4[th] order system previously tackled by Parthasarathy et al. [13] :

$$G_4(s) = \frac{248s^2 + 900s + 1200}{s^4 + 18s^3 + 102s^2 + 180s + 120}$$

The poles of the above system are given by :

$-1.1967 \pm 0.6934i,\ -7.8033 \pm 1.3576i$.

If a 2[nd] order model is desired, then the steps to be followed are as under :

Step-1: Selection of dominant poles to be retained :

$\lambda_1 = -1.1967 + 0.6934i$, $\lambda_2 = -1.1967 - 0.6934i$ are the poles to be retained. Therefore,

$$D_2(s) = s^2 + 2.3934s + 1.9129 \qquad (12)$$

Step-2 : Evaluate the MCF quotients by forming the array

$h_1 = 0.1$	120	180	102	18	1	
	1200	900	248	0		$H_1 = 0$
$h_2 = 0.075$	90	77.2	18	1		
	1200		248			$H_2 = 248$
			1			

Step-3 : Construct the modified Routh array as in (5)

$h_1 = 0.1$	1.9129	2.3934	1	
	19.129	0		$H_1 = 0$
		1		

(14)

Therefore, the reduced 2[nd] order model is given by :

$$G_2(s) = \frac{19.129}{s^2 + 2.3934s + 1.9129} \qquad (15)$$

with an I.S.E. of 11.448115.

The step response of original and reduced order models is shown in Figure 1 and a comparison of the proposed method with Parthasarathy et al. [13] is given in Table I.

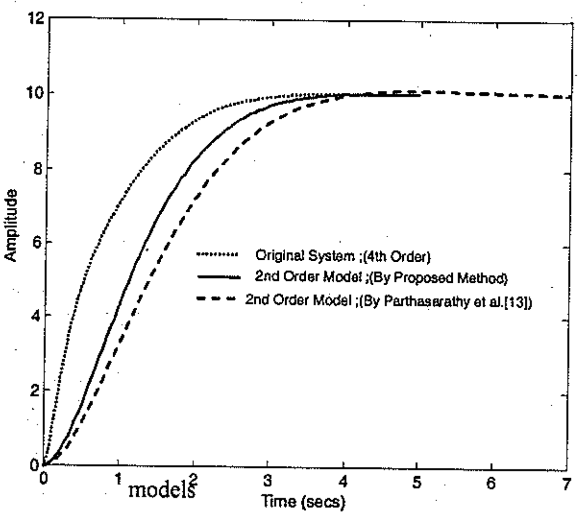


Table I
Comparison of reduced order models

Method of order reduction	Reduced models, $[G_2(s)]$	I.S.E.
Proposed method	$\frac{19.129}{s^2+2.3934s+1.9129}$	11.448115
Parthasarathy et al. [13]	$\frac{11.9}{s^2+1.785s+1.19}$	22.419383

Example-2. Consider a 4[th] order system previously tackled by Mukherjee et al. [15] :

$$G_4(s) = \frac{s^3+7s^2+24s+24}{s^4+10s^3+35s^2+50s+24}$$

The poles of the above system are all real and given by : $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -4$.

By using the proposed method, the following 2[nd] and 3[rd] order approximants are obtained :

$$G_2(s) = \frac{s+2}{s^2+3s+2} \qquad (16)$$

with an I.S.E. of 3.57135×10^{-3}.

$$G_3(s) = \frac{s^2+4.5s+6}{s^3+6s^2+11s+6} \qquad (17)$$

with an I.S.E. of 0.892614×10^{-3}.

The step response of original and reduced order models is shown in Figure 2 and a comparison of the proposed method with other methods, for a 2[nd] order reduced model, ($G_2(s)$) is given in Table II.

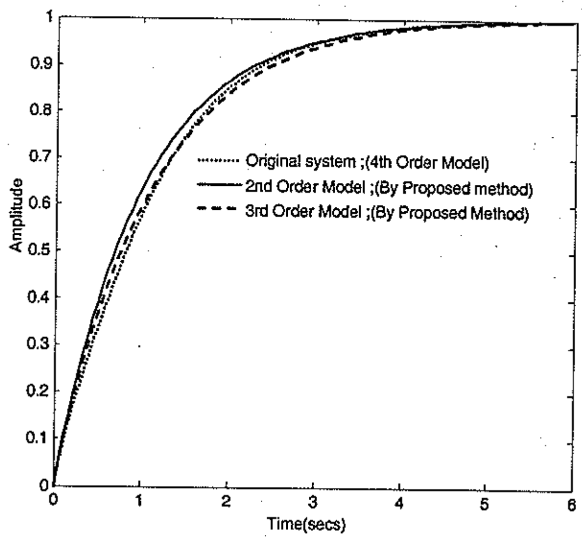


Figure 2. Step responses of original and reduced order models.

Table II

Comparison of reduced order models

Method of order reduction	Reduced 2[nd] order models, $[G_2(s)]$	I.S.E.
Proposed method	$\frac{s+2}{s^2+3s+2}$	3.57135×10^{-3}
Shieh & Wei [17]	$\frac{s+2.3014}{s^2+5.7946s+2.3014}$	142.5607×10^{-3}
Krishnamurthy et al. [18]	$\frac{20.5714s+24}{30s^2+42s+24}$	9.5891×10^{-3}
J. Pal [19]	$\frac{16.0008s+24}{30s^2+42s+24}$	1.1688×10^{-2}
Gutman et al. [20]	$\frac{2[48s+144]}{70s^2+300s+288}$	4.5593×10^{-2}
Prasad et al. [21]	$\frac{s+34.2465}{s^2+239.8082s+34.2465}$	1.53427

Example-3. Consider a 6th order two input two output system [9] described by the transfer matrix :

$$[G(s)] = \begin{bmatrix} \dfrac{2(s+5)}{(s+1)(s+10)} & \dfrac{(s+4)}{(s+2)(s+5)} \\ \dfrac{(s+10)}{(s+1)(s+20)} & \dfrac{(s+6)}{(s+2)(s+3)} \end{bmatrix}$$

$$= \frac{1}{D_6(s)} \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix} \qquad (18)$$

where, the common denominator $D_6(s)$ is given by :

$$D_6(s) = (s+1)(s+2)(s+3)(s+5)(s+10)(s+20)$$
$$= 6000 + 13100s + 10060s^2 + 3491s^3 + 571s^4 + 41s^5 + s^6$$

and

$$a_{11}(s) = 6000 + 7700s + 3610s^2 + 762s^3 + 70s^4 + 2s^5$$
$$a_{12}(s) = 2400 + 4160s + 2182s^2 + 459s^3 + 38s^4 + s^5$$
$$a_{21}(s) = 3000 + 3700s + 1650s^2 + 331s^3 + 30s^4 + s^5$$
$$a_{22}(s) = 6000 + 9100s + 3660s^2 + 601s^3 + 42s^4 + s^5$$

The proposed method is successively applied to each element of the transfer matrix of above multivariable system and the reduced order models $r_{ij}(s)$ of the LOS [R(s)] are obtained. The general form of second order model is taken as :

$$[R(s)] = \frac{1}{D_2(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) \\ b_{21}(s) & b_{22}(s) \end{bmatrix} \qquad (18)$$

where, $D_2(s) = s^2 + 3s + 2$.

and $b_{11}(s) = 2s + 2$, $b_{12}(s) = s + 0.8$

$b_{21}(s) = s + 1$, $b_{22}(s) = s + 2$

The step responses of original and reduced order models are compared in Figure3 (a-d) and a comparison of the proposed method with Prasad et al. [9] is given in Table III.

Table III

I.S.E. for Reduced Order Models

r_{ij}	I.S.E. (By Proposed Method)	I.S.E. (By Prasad & Pal [9])
r_{11}	0.051618	0.135505
r_{12}	0.001986	0.037593
r_{21}	0.016543	0.040013
r_{22}	0.032249	0.067897

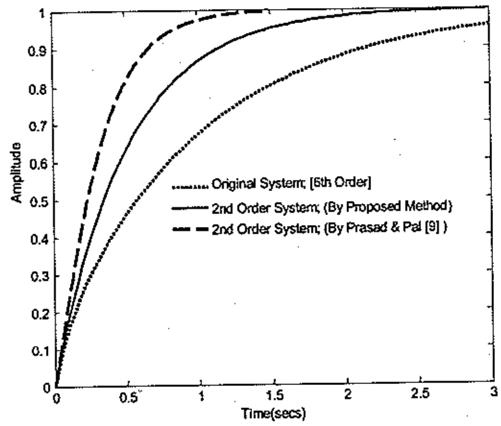


Figure 3(a) Comparison of Step Responses; u_1=1, u_2 = 0

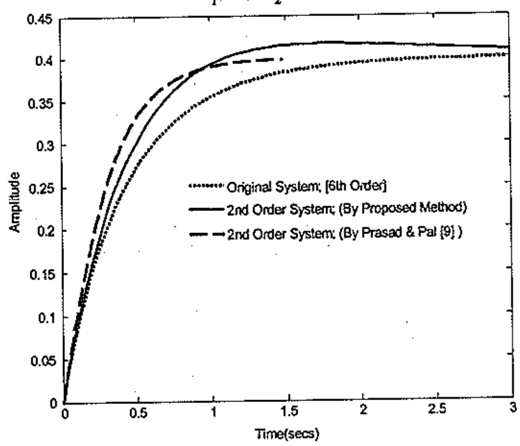


Figure 3(b) Comparison of Step Responses; u_1 = 0, u_2 = 1

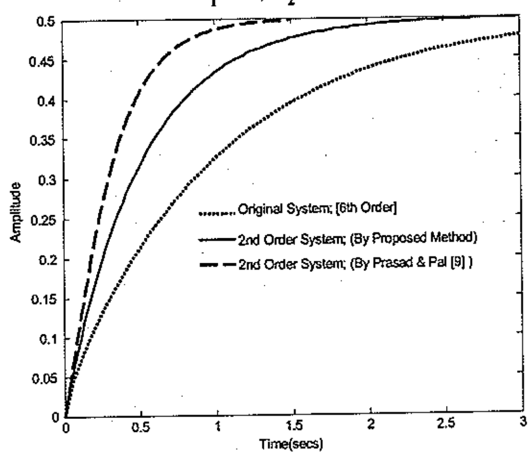


Figure 3(c).Comparison of Step Responses; u_1 = 1, u_2 = 0

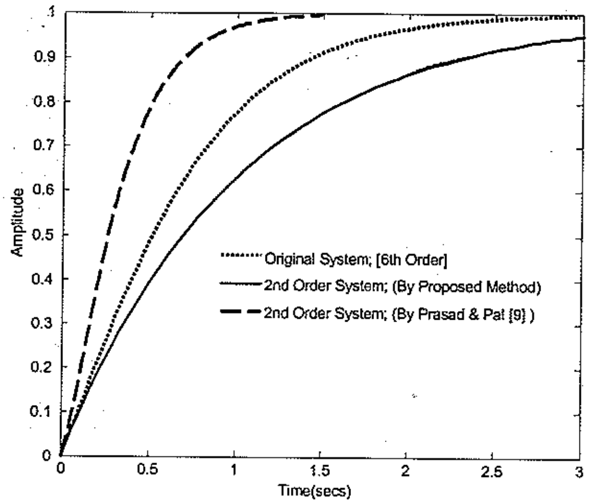


Figure 3(d). Comparison of Step Responses; $u_1 = 0, u_2 = 1$

5. Conclusions

An algorithm, which combines the advantages of the dominant pole retention method and the modified Cauer continued fraction, has been presented, to derive stable reduced order models for linear dynamic systems.

In this method the dominant poles are retained according to the order to be reduced to, and zeros are synthesized by using the modified Cauer continued fraction technique. The method has also been extended for order reduction of linear multivariable systems. The method is simple, rugged and computer oriented. The matching of the step response is assured reasonably well in the method. The I.S.E. in between the transient parts of original and reduced order systems is calculated from which it is clear that the proposed method compares well with the other techniques of model order reduction as shown in Tables I, II and III. The method preserves model stability and avoids any error in between the initial or final values of the responses of original and reduced order models.

References

[1] R. Genesio and M. Milanese, *"A note on the derivation and use of reduced order models"*, IEEE Trans. Automat Control, Vol. AC-21, No. 1, pp. 118-122, February 1976.

[2] M. S. Mahmoud and M. G. Singh, *"Large scale systems modelling,"* 1981, Pergamon Press, International Series on Systems and Control 1st ed., Vol. 3.

[3] M. Jamshidi, *"Large scale systems modelling and control series,"* New York, 1983, Amsterdam, Oxford, North Holland, Vol. 9.

[4] S. K. Nagar and S. K. Singh, *"An algorithmic approach for system decomposition and balanced realized model reduction"*, *Journal of Franklin Inst.*, Vol. 341, pp. 615-630, 2004.

[5] V. Singh, D. Chandra and H. Kar, *"Improved Routh Pade approximants : A computer aided approach"*, IEEE Trans. Automat Control, Vol. 49, No.2, pp 292-296, February 2004.

[6] S. Mukherjee, Satakshi and R.C.Mittal, *"Model order reduction using response-matching technique"*, Journal of Franklin Inst., Vol. 342 , pp. 503-519, 2005.

[7] S. Mukherjee and R.N. Mishra, *"Reduced order modelling of linear multivariable systems using an error minimization technique"*, Journal of Franklin Inst., Vol. 325, No.2 , pp. 235-245, 1988.

[8] S. S. Lamba, R. Gorez and B. Bandyopadhyay, *"New reduction technique by step error minimization for multivariable systems"*, Int. J. Systems Sci., Vol.19, No.6, pp. 999-1009, 1988.

[9] R. Prasad and J. Pal, *"Use of continued fraction expansion for stable reduction of linear multivariable*

systems", Journal of Institution of Engineers IE(I) Journal – EL, Vol. 72, pp. 43-47, June 1991.

[10] R. Prasad, *"Pade type model order reduction for multivariable systems using Routh approximation"*, Computers and Electrical Engineering, Vol. 26, pp. 445-459, 2000.

[11] R. Parthasarathy and S. John, *"System reduction by Routh approximation and modified Cauer continued fraction"*, Electronic Letters, Vol. 15, pp. 691-692, 1979.

[12] T.C. Chen, C.Y. Chang and K.W. Han, *"Model reduction using the stability equation method and the continued fraction method"*, Int. J. Control, Vol. 32, pp. 81-94, 1980.

[13] R. Parthasarathy and K.N. Jayasimha, *"System reduction using stability equation method and modified Cauer continued fraction"*, Proceedings IEEE, Vol. 70, No. 10, pp.1234-1236, October 1982.

[14] E. J. Davison, *"A method for simplifying linear dynamic system"*, IEEE Trans. Automat. Control, AC-11, pp. 93-101, 1966.

[15] S. Mukherjee and R. N. Mishra, *"Order reduction of linear systems using an error minimization technique"*, Journal of Franklin Institute, Vol. 323, No. 1, pp. 23-32, 1987.

[16] R. Parthasarathy and S. John, *"State space models using modified Cauer continued fraction"*, Proceedings IEEE (lett.), Vol. 70, No. 3, pp.300-301, March 1982.

[17] L.S. Shieh and Y.J. Wei, *"A mixed method for multivariable system reduction"*, IEEE Trans. Automat. Control, Vol. AC-20, pp. 429-432, 1975.

[18] V. Krishnamurthy and V. Seshadri, *"Model reduction using the Routh stability criterion"*, IEEE Trans. Automat. Control, Vol. AC-23, No.4, pp. 729-731, August 1978.

[19] J. Pal, April 1979, *"Stable reduced order Pade approximants using the Routh Hurwitz array"*, Electronic Letters, Vol. 15, No.8, pp.225-226, April 1979.

[20] P. Gutman, C.F. Mannerfelt and P. Molander, *"Contributions to the model reduction problem"*, IEEE Trans. Automat. Control, Vol. AC-27, No.2, pp. 454-455, April 1982.

[21] R. Prasad and J. Pal, *"Stable reduction of linear systems by continued fractions"*, Journal of Institution of Engineers IE(I) Journal – EL, Vol. 72, pp. 113-116, October 1991.

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